UNIT 2
Congruence

Focus
Use a variety of representations, tools, and technology to solve meaningful problems by representing and transforming figures and analyzing relationships.

CHAPTER 4
Congruent Triangles
BIG Idea Analyze geometric relationships in order to make and verify conjectures involving triangles.
BIG Idea Apply the concept of congruence to justify properties of figures and solve problems.

CHAPTER 5
Relationships in Triangles
BIG Idea Use a variety of representations to describe geometric relationships and solve problems involving triangles.

CHAPTER 6
Quadrilaterals
BIG Idea Analyze properties and describe relationships in quadrilaterals.
BIG Idea Apply logical reasoning to justify and prove mathematical statements involving quadrilaterals.
Geometry and History

Who is behind this geometry idea anyway? Have you ever wondered who first developed some of the ideas you are learning in your geometry class? Many ideas we study were developed many years ago, but people today are also discovering new mathematics. Mathematicians continue to study fractals that were pioneered by Benoit Mandelbrot and Gaston Julia. In this project, you will be using the Internet to research a topic in geometry. You will then prepare a portfolio or poster to display your findings.

Log on to geometryonline.com to begin.
Foldables
Study Organizer

**Congruent Triangles** Make this Foldable to help you organize your notes. Begin with two sheets of grid paper and one sheet of construction paper.

1. **Stack** the grid paper on the construction paper. Fold diagonally to form a triangle and cut off the excess.

2. **Staple** the edge to form a booklet. Write the chapter title on the front and label each page with a lesson number and title.

**Real-World Link**

**Triangles** Triangles with the same size and shape can be modeled by a pair of butterfly wings.

**Key Vocabulary**
- exterior angle (p. 211)
- flow proof (p. 212)
- corollary (p. 213)
- congruent triangles (p. 217)
- coordinate proof (p. 251)

**Big Ideas**
- Classify triangles.
- Apply the Angle Sum Theorem and the Exterior Angle Theorem.
- Identify corresponding parts of congruent triangles.
- Test for triangle congruence using SSS, SAS, ASA, and AAS.
- Use properties of isosceles and equilateral triangles.
- Write coordinate proofs.
Option 1

Take the Quick Check below. Refer to the Quick Review for help.

Solve each equation. (Prerequisite Skill)

1. \(2x + 18 = 5\)
2. \(3m - 16 = 12\)
3. \(6 = 2a + \frac{1}{2}\)
4. \(\frac{2}{3}b + 9 = -15\)
5. FISH Miranda bought 4 goldfish and $5 worth of accessories. She spent a total of $6 at the store. Write and solve an equation to find the amount for each goldfish. (Prerequisite Skill)

EXAMPLE 1 Solve \(\frac{7}{8}t + 4 = 18\).

\[
\frac{7}{8}t + 4 = 18
\]
Write the equation.

\[
\frac{7}{8}t = 14
\]
Subtract.

\[
8\left(\frac{7}{8}\right) = 14(8)
\]
Multiply.

\[
7t = 112
\]
Simplify.

\[
t = 16
\]
Divide each side by 7.

EXAMPLE 2 Name the angles congruent to \(\angle 6\) if \(a \parallel b\).

\[\angle 8 \equiv \angle 6\] Vertical Angle Theorem

\[\angle 2 \equiv \angle 6\] Corresponding Angles Postulate

\[\angle 4 \equiv \angle 6\] Alternate Exterior Angles Theorem

EXAMPLE 3 Find the distance between \((-1, 2)\) and \((3, -4)\). Round to the nearest tenth.

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]
Distance Formula

\[
d = \sqrt{(3 - (-1))^2 + (-4 - 2)^2}
\]
\[
d = \sqrt{4^2 + (-6)^2}
\]
Subtract.

\[
d = \sqrt{16 + 36}
\]
Simplify.

\[
d = \sqrt{52}
\]
Add.

\[
d \approx 7.2
\]
Use a calculator.

Name the indicated angles or pairs of angles if \(p \parallel q\) and \(m \parallel \ell\). (Lesson 3-1)

6. angles congruent to \(\angle 8\)
7. angles supplementary to \(\angle 12\)

Find the distance between each pair of points. Round to the nearest tenth. (Lesson 1-3)

8. \((6, 8), (-4, 3)\)
9. \((11, -8), (-3, -4)\)

10. MAPS Jack laid a coordinate grid on a map where each block on the grid corresponds to a city block. If the coordinates of the football stadium are \((15, -25)\) and the coordinates of Jack’s house are \((-8, 14)\), what is the distance between the stadium and Jack’s house? Round to the nearest tenth. (Lesson 1-3)
Many structures use triangular shapes as braces for construction. The roof sections of houses are made of triangular trusses that support the roof and the house.

Classify Triangles by Angles Triangle $ABC$, written $\triangle ABC$, has parts that are named using the letters $A$, $B$, and $C$.

- The sides of $\triangle ABC$ are $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$.
- The vertices are $A$, $B$, and $C$.
- The angles are $\angle ABC$ or $\angle B$, $\angle BCA$ or $\angle C$, and $\angle BAC$ or $\angle A$.

There are two ways to classify triangles. One way is by their angles. All triangles have at least two acute angles, but the third angle is used to classify the triangle.

<table>
<thead>
<tr>
<th>Classification</th>
<th>Acute Triangle</th>
<th>Obtuse Triangle</th>
<th>Right Triangle</th>
</tr>
</thead>
<tbody>
<tr>
<td>Description</td>
<td>All angles are acute.</td>
<td>One angle is obtuse.</td>
<td>One angle is right.</td>
</tr>
<tr>
<td>Angles</td>
<td>$37^\circ$, $67^\circ$, $76^\circ$</td>
<td>$13^\circ$, $142^\circ$, $25^\circ$</td>
<td>$42^\circ$, $90^\circ$, $48^\circ$</td>
</tr>
<tr>
<td>All angles</td>
<td>All angles measure $&lt; 90$.</td>
<td>One angle measure $&gt; 90$.</td>
<td>One angle measure $= 90$.</td>
</tr>
</tbody>
</table>

An acute triangle with all angles congruent is an equiangular triangle.
Classify Triangles by Angles

ARCHITECTURE  The roof of this house is made up of three different triangles. Use a protractor to classify \( \triangle DFH \), \( \triangle DFG \), and \( \triangle HFG \) as acute, equiangular, obtuse, or right.

\( \triangle DFH \) has all angles with measures less than 90, so it is an acute triangle. \( \triangle DFG \) and \( \triangle HFG \) both have one angle with measure equal to 90. Both of these are right triangles.

1. BICYCLES  The frame of this tandem bicycle uses triangles. Use a protractor to classify \( \triangle ABC \) and \( \triangle CDE \).

Classify Triangles by Sides  Triangles can also be classified according to the number of congruent sides they have.

<table>
<thead>
<tr>
<th>No two sides of a <strong>scalene triangle</strong> are congruent.</th>
<th>At least two sides of an <strong>isosceles triangle</strong> are congruent.</th>
<th>All of the sides of an <strong>equilateral triangle</strong> are congruent.</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Scalene Triangle" /></td>
<td><img src="image2" alt="Isosceles Triangle" /></td>
<td><img src="image3" alt="Equilateral Triangle" /></td>
</tr>
</tbody>
</table>

Equilateral Triangles

An equilateral triangle is a special kind of isosceles triangle.

Congruency

To indicate that sides of a triangle are congruent, an equal number of hash marks are drawn on the corresponding sides.

Study Tip

**Extra Examples at:** geometryonline.com

GEOMETRY LAB

**Equilateral Triangles**

**MODEL**

- Align three pieces of patty paper. Draw a dot at \( X \).
- Fold the patty paper through \( X \) and \( Y \) and through \( X \) and \( Z \).

**ANALYZE**

1. Is \( \triangle XYZ \) equilateral? Explain.
2. Use three pieces of patty paper to make a triangle that is isosceles, but not equilateral.
3. Use three pieces of patty paper to make a scalene triangle.
EXAMPLE
Classify Triangles by Sides

Identify the indicated type of triangle in the figure.

a. isosceles triangles
   Isosceles triangles have at least two sides congruent.
   So, \( \triangle ABD \) and \( \triangle EBD \) are isosceles.

b. scalene triangles
   Scalene triangles have no congruent sides.
   \( \triangle AEB, \triangle AED, \triangle ACB, \)
   \( \triangle ACD, \triangle BCE, \) and \( \triangle DCE \) are scalene.

CHECK Your Progress

Identify the indicated type of triangle in the figure.

2A. equilateral
2B. isosceles

EXAMPLE
Find Missing Values

ALGEBRA
Find \( x \) and the measure of each side of equilateral triangle \( RST \).

Since \( \triangle RST \) is equilateral, \( RS = ST \).

\[
x + 9 = 2x \quad \text{Substitution}
\]
\[
9 = x \quad \text{Subtract} \ x \text{ from each side.}
\]

Next, substitute to find the length of each side.

\[
RS = x + 9 \quad ST = 2x \quad RT = 3x - 9
\]
\[
= 9 + 9 \text{ or } 18 \quad = 2(9) \text{ or } 18 \quad = 3(9) - 9 \text{ or } 18
\]

For \( \triangle RST \), \( x = 9 \), and the measure of each side is 18.

CHECK Your Progress

3. Find \( x \) and the measure of the unknown sides of isosceles triangle \( EFG \).

EXAMPLE
Use the Distance Formula

COORDINATE GEOMETRY
Find the measures of the sides of \( \triangle DEC \). Classify the triangle by sides.

Use the Distance Formula to find the lengths of each side.

\[
EC = \sqrt{(-5 - 2)^2 + (3 - 2)^2}
\]
\[
= \sqrt{49 + 1}
\]
\[
= \sqrt{50} \text{ or } 5\sqrt{2}
\]
Use a protractor to classify each triangle as \textit{acute, equiangular, obtuse,} or \textit{right}.

1. 

2. 

Identify the indicated type of triangle in the figure.

3. \textit{isosceles} 

4. \textit{scalene} 

\textbf{ALGEBRA} Find \(x\) and the measures of the unknown sides of each triangle.

5. 

6. 

7. \textbf{COORDINATE GEOMETRY} Find the measures of the sides of \(\triangle TWZ\) with vertices at \(T(2, 6), W(4, -5),\) and \(Z(-3, 0)\). Classify the triangle by sides.

8. \textbf{COORDINATE GEOMETRY} Find the measures of the sides of \(\triangle QRS\) with vertices at \(Q(2, 1), R(4, -3),\) and \(S(-3, -2)\). Classify the triangle by sides.

\[
DC = \sqrt{(3 - 2)^2 + (9 - 2)^2} = \sqrt{1 + 49} = \sqrt{50} \text{ or } 5\sqrt{2}
\]

\[
ED = \sqrt{(-5 - 3)^2 + (3 - 9)^2} = \sqrt{64 + 36} = \sqrt{100} \text{ or } 10
\]

Since \(\overline{EC}\) and \(\overline{DC}\) have the same length, \(\triangle DEC\) is \textit{isosceles}.
13. Identify the obtuse triangles if $\angle MJK \cong \angle KLM$, $m\angle MJK = 126$, and $m\angle JNM = 52$.

14. Identify the right triangles if $IJ \parallel GH$, $GH \perp DF$, and $GI \perp EF$.

15. **ALGEBRA** Find $x$, $JM$, $MN$, and $JN$ if $\triangle JMN$ is an isosceles triangle with $JM \cong MN$.

16. **ALGEBRA** Find $x$, $QR$, $RS$, and $QS$ if $\triangle QRS$ is an equilateral triangle.

**COORDINATE GEOMETRY** Find the measures of the sides of $\triangle ABC$ and classify each triangle by its sides.

17. $A(5, 4), B(3, -1), C(7, -1)$
18. $A(-4, 1), B(5, 6), C(-3, -7)$
19. $A(-7, 9), B(-7, -1), C(4, -1)$
20. $A(-3, -1), B(2, 1), C(2, -3)$

21. **QUILTING** The star-shaped composite quilting square is made up of four different triangles. Use a ruler to classify the four triangles by sides.

22. **ARCHITECTURE** The restored and decorated Victorian houses in San Francisco shown in the photograph are called the “Painted Ladies.” Use a protractor to classify the triangles indicated in the photo by sides and angles.

Identify the indicated triangles in the figure if $AB \cong BD \cong DC \cong CA$ and $BC \perp AD$.

23. right
24. obtuse
25. scalene
26. isosceles

27. **ASTRONOMY** On May 5, 2002, Venus, Saturn, and Mars were aligned in a triangular formation. Use a protractor or ruler to classify the triangle formed by sides and angles.

28. **RESEARCH** Use the Internet or other resource to find out how astronomers can predict planetary alignment.

Real-World Link

The Painted Ladies are located in Alamo Square. The area is one of 11 designated historic districts in San Francisco.

Source: [www.sfvisitor.org](http://www.sfvisitor.org)
**Lesson 4-1**

### Classifying Triangles

#### Algebra

**29.** \( \triangle GHJ \) is isosceles, with \( \overline{HG} \equiv \overline{GJ} \), \( GH = x + 7 \), \( GJ = 3x - 5 \), and \( HJ = x - 1 \).

**30.** \( \triangle MPN \) is equilateral with \( MN = 3x - 6 \), \( MP = x + 4 \), and \( NP = 2x - 1 \).

**31.** \( \triangle QRS \) is equilateral. \( QR \) is two less than two times a number, \( RS \) is six more than the number, and \( QS \) is ten less than three times the number.

**32.** \( \triangle JKL \) is isosceles with \( \overline{KJ} \equiv \overline{LJ} \). \( JL \) is five less than two times a number. \( JK \) is three more than the number. \( KL \) is one less than the number. Find the measure of each side.

**33.** **Road Trip** The total distance from Charlotte to Raleigh to Winston-Salem and back to Charlotte is about 292 miles. The distance from Charlotte to Winston-Salem is 22 miles less than the distance from Raleigh to Winston-Salem. The distance from Charlotte to Raleigh is 60 miles greater than the distance from Winston-Salem to Charlotte. Classify the triangle that connects Charlotte, Raleigh, and Winston-Salem.

**34.** **Crystal** The top of the crystal bowl pictured at the right is circular. The diameter at the top of the bowl is \( \overline{MN} \). \( P \) is the midpoint of \( MN \), and \( OP \perp MN \). If \( MN = 24 \) and \( OP = 12 \), determine whether \( \triangle MPO \) and \( \triangle NPO \) are equilateral.

**35.** **Proof** Write a two-column proof to prove that \( \triangle EQL \) is equiangular.

**36.** **Proof** Write a paragraph proof to prove that \( \triangle RPM \) is an obtuse triangle if \( m\angle NPM = 33 \).

**37.** **Coordinate Geometry** Show that \( S \) is the midpoint of \( \overline{RT} \) and \( U \) is the midpoint of \( \overline{TV} \).

**38.** **Coordinate Geometry** Show that \( \triangle ADC \) is isosceles.

**39.** **Open Ended** Draw an isosceles right triangle.

**Reasoning** Determine whether each statement is **always**, **sometimes**, or **never** true. Explain.

**40.** Equiangular triangles are also acute. **41.** Right triangles are acute.
42. **CHALLENGE** \( \overline{KL} \) is a segment representing one side of isosceles right triangle \( KLM \) with \( K(2, 6) \), and \( L(4, 2) \). \( \angle KLM \) is a right angle, and \( \overline{KL} \cong \overline{LM} \). Describe how to find the coordinates of \( M \) and name these coordinates.

43. **Writing in Math** Use the information on page 202 to explain why triangles are important in construction. Include a description of how to classify triangles and a justification of why you think one type of triangle might be used more often in architecture than other types.

44. Which type of triangle can serve as a counterexample to the conjecture below?

If two angles of a triangle are acute, then the measure of the third angle must be greater than or equal to 90.

- A equilateral
- B obtuse
- C right
- D scalene

45. A baseball glove originally cost $84.50. Jamal bought it at 40% off. How much was deducted from the original price?

- F $50.70
- H $33.80
- G $44.50
- J $32.62

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**Spiral Review**

Graph each line. Construct a perpendicular segment through the given point. Then find the distance from the point to the line. (Lesson 3-6)

46. \( y = x + 2, (2, -2) \)
47. \( x + y = 2, (3, 3) \)
48. \( y = 7, (6, -2) \)

Find \( x \) so that \( p \parallel q \). (Lesson 3-5)

49. \( 110^\circ \) \( 110^\circ \)
50. \( (3x - 50)^\circ \) \( (2x - 5)^\circ \)
51. \( 57^\circ \) \( (3x - 9)^\circ \)

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**PREREQUISITE SKILL** In the figure, \( \overline{AB} \parallel \overline{RQ} \), \( \overline{BC} \parallel \overline{PR} \), and \( \overline{AC} \parallel \overline{PQ} \). Name the indicated angles or pairs of angles. (Lessons 3-1 and 3-2)

52. three pairs of alternate interior angles
53. six pairs of corresponding angles
54. all angles congruent to \( \angle 3 \)
55. all angles congruent to \( \angle 7 \)
56. all angles congruent to \( \angle 11 \)

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**EXPLORE 4-2**

Geometry Lab

**Angles of Triangles**

**ACTIVITY 1**

Find the relationship among the measures of the interior angles of a triangle.

**Step 1** Draw an obtuse triangle and cut it out. Label the vertices \(A\), \(B\), and \(C\).

**Step 2** Find the midpoint of \(AB\) by matching \(A\) to \(B\). Label this point \(D\).

**Step 3** Find the midpoint of \(BC\) by matching \(B\) to \(C\). Label this point \(E\).

**Step 4** Draw \(DE\).

**Step 5** Fold \(\triangle ABC\) along \(DE\). Label the point where \(B\) touches \(AC\) as \(F\).

**Step 6** Draw \(DF\) and \(FE\). Measure each angle.

**ANALYZE THE MODEL**

Describe the relationship between each pair.

1. \(\angle A\) and \(\angle DFA\)
2. \(\angle B\) and \(\angle DFE\)
3. \(\angle C\) and \(\angle EFC\)
4. What is the sum of the measures of \(\angle DFA\), \(\angle DFE\), and \(\angle EFC\)?
5. What is the sum of the measures of \(\angle A\), \(\angle B\), and \(\angle C\)?
6. **Make a conjecture** about the sum of the measures of the angles of any triangle.

In the figure at the right, \(\angle 4\) is called an **exterior angle** of the triangle. \(\angle 1\) and \(\angle 2\) are the **remote interior angles** of \(\angle 4\).

**ACTIVITY 2**

Find the relationship among the interior and exterior angles of a triangle.

**Step 1** Trace \(\triangle ABC\) from Activity 1 onto a piece of paper. Label the vertices.

**Step 2** Extend \(AC\) to draw an exterior angle at \(C\).

**Step 3** Tear \(\angle A\) and \(\angle B\) off the triangle from Activity 1.

**Step 4** Place \(\angle A\) and \(\angle B\) over the exterior angle.

**ANALYZE THE RESULTS**

7. **Make a conjecture** about the relationship of \(\angle A\), \(\angle B\), and the exterior angle at \(C\).
8. Repeat the steps for the exterior angles of \(\angle A\) and \(\angle B\).
9. Is your conjecture true for all exterior angles of a triangle?
10. Repeat Activity 2 with an acute triangle and with a right triangle.
11. **Make a conjecture** about the measure of an exterior angle and the sum of the measures of its remote interior angles.
Angles of Triangles

Main Ideas
- Apply the Angle Sum Theorem.
- Apply the Exterior Angle Theorem.

New Vocabulary
exterior angle
remote interior angles
flow proof
corollary

The Drachen Foundation coordinates the annual Miniature Kite Contest. In a recent year, the kite in the photograph won second place in the Most Beautiful Kite category. The overall dimensions are 10.5 centimeters by 9.5 centimeters. The wings of the beetle are triangular.

Angle Sum Theorem If the measures of two of the angles of a triangle are known, how can the measure of the third angle be determined? The Angle Sum Theorem explains that the sum of the measures of the angles of any triangle is always 180.

**Theorem 4.1 Angle Sum**

The sum of the measures of the angles of a triangle is 180.

Example: \( m\angle W + m\angle X + m\angle Y = 180 \)

**Proof** Angle Sum Theorem

**Given:** \( \triangle ABC \)

**Prove:** \( m\angle C + m\angle 2 + m\angle B = 180 \)

**Proof:**

<table>
<thead>
<tr>
<th>Statements</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \triangle ABC )</td>
</tr>
<tr>
<td>2. Draw ( \overline{XY} ) through ( A ) parallel to ( \overline{CB} ).</td>
</tr>
<tr>
<td>3. ( \angle 1 ) and ( \angle CAY ) form a linear pair.</td>
</tr>
<tr>
<td>4. ( \angle 1 ) and ( \angle CAY ) are supplementary.</td>
</tr>
<tr>
<td>5. ( m\angle 1 + m\angle CAY = 180 )</td>
</tr>
<tr>
<td>6. ( m\angle CAY = m\angle 2 + m\angle 3 )</td>
</tr>
<tr>
<td>7. ( m\angle 1 + m\angle 2 + m\angle 3 = 180 )</td>
</tr>
<tr>
<td>8. ( \angle 1 \cong \angle C, \angle 3 \cong \angle B )</td>
</tr>
<tr>
<td>9. ( m\angle 1 = m\angle C, m\angle 3 = m\angle B )</td>
</tr>
<tr>
<td>10. ( m\angle C + m\angle 2 + m\angle B = 180 )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Given</td>
</tr>
<tr>
<td>2. Parallel Postulate</td>
</tr>
<tr>
<td>3. Def. of a linear pair</td>
</tr>
<tr>
<td>4. If 2 ( \triangle ) form a linear pair, they are supplementary.</td>
</tr>
<tr>
<td>5. Def. of suppl. ( \triangle )</td>
</tr>
<tr>
<td>6. Angle Addition Postulate</td>
</tr>
<tr>
<td>7. Substitution</td>
</tr>
<tr>
<td>8. Alt. Int. ( \triangle ) Theorem</td>
</tr>
<tr>
<td>9. Def. of ( \cong \triangle )</td>
</tr>
<tr>
<td>10. Substitution</td>
</tr>
</tbody>
</table>

**Auxiliary Lines**
Recall that sometimes extra lines have to be drawn to complete a proof. These are called auxiliary lines.

Study Tip
If we know the measures of two angles of a triangle, we can find the measure of the third.

**EXAMPLE**  **Interior Angles**

Find the missing angle measures.

Find $m\angle 1$ first because the measures of two angles of the triangle are known.

\[
m\angle 1 + 28 + 82 = 180 \quad \text{Angle Sum Theorem}
\]
\[
m\angle 1 + 110 = 180 \quad \text{Simplify.}
\]
\[
m\angle 1 = 70 \quad \text{Subtract 110 from each side.}
\]

$\angle 1$ and $\angle 2$ are congruent vertical angles. So $m\angle 2 = 70$.

\[
m\angle 3 + 68 + 70 = 180 \quad \text{Angle Sum Theorem}
\]
\[
m\angle 3 + 138 = 180 \quad \text{Simplify.}
\]
\[
m\angle 3 = 42 \quad \text{Subtract 138 from each side.}
\]

Therefore, $m\angle 1 = 70$, $m\angle 2 = 70$, and $m\angle 3 = 42$.

The Angle Sum Theorem leads to a useful theorem about the angles in two triangles.

**THEOREM 4.2**  **Third Angle Theorem**

If two angles of one triangle are congruent to two angles of a second triangle, then the third angles of the triangles are congruent.

**Example:** If $\angle A \cong \angle F$ and $\angle C \cong \angle D$, then $\angle B \cong \angle E$.

You will prove this theorem in Exercise 34.

**Exterior Angle Theorem**  Each angle of a triangle has an exterior angle. An **exterior angle** is formed by one side of a triangle and the extension of another side. The interior angles of the triangle not adjacent to a given exterior angle are called **remote interior angles** of the exterior angle.
We will use a flow proof to prove this theorem. A **flow proof** organizes a series of statements in logical order, starting with the given statements. Each statement is written in a box with the reason verifying the statement written below the box. Arrows are used to indicate how the statements relate to each other.

**Flow Proof**

Write each statement and reason on an index card. Then organize the index cards in logical order.

**Proof**

Write a flow proof of the Exterior Angle Theorem.

**Given:** \( \triangle ABC \)

**Prove:** \( m\angle CBD = m\angle A + m\angle C \)

**Flow Proof:**

1. \( \triangle ABC \)  
   - Given
2. \( \angle CBD \) and \( \angle ABC \) form a linear pair.  
   - Definition of linear pair
3. \( \angle CBD \) and \( \angle ABC \) are supplementary.  
   - If 2 \( \triangle \) form a linear pair, they are supplementary.
4. \( m\angle A + m\angle ABC + m\angle C = 180 \)  
   - Angle Sum Theorem
5. \( m\angle CBD + m\angle ABC = 180 \)  
   - Definition of supplementary
6. \( m\angle A + m\angle ABC + m\angle C = m\angle CBD + m\angle ABC \)  
   - Substitution Property
7. \( m\angle A + m\angle C = m\angle CBD \)  
   - Subtraction Property

**Example**

Find the measure of each angle.

a. \( m\angle 1 \)
   \[
   m\angle 1 = 50 + 78 \quad \text{Exterior Angle Theorem} \\
   = 128 \quad \text{Simplify.}
   \]

b. \( m\angle 2 \)
   \[
   m\angle 1 + m\angle 2 = 180 \quad \text{If 2 \( \triangle \) form a linear pair, they are suppl.} \\
   128 + m\angle 2 = 180 \quad \text{Substitution} \\
   m\angle 2 = 52 \quad \text{Subtract 128 from each side.}
   \]
c. \( m\angle 3 \)
\[
\begin{align*}
\angle 2 + m\angle 3 &= 120 & \text{Exterior Angle Theorem} \\
52 + m\angle 3 &= 120 & \text{Substitution} \\
m\angle 3 &= 68 & \text{Subtract 52 from each side.}
\end{align*}
\]
Therefore, \( m\angle 1 = 128 \), \( m\angle 2 = 52 \), and \( m\angle 3 = 68 \).

2A. \( m\angle 4 \)  
2B. \( m\angle 5 \)

A statement that can be easily proved using a theorem is often called a corollary of that theorem. A corollary, just like a theorem, can be used as a reason in a proof.

### COROLLARIES

<table>
<thead>
<tr>
<th>Corollary</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>The acute angles of a right triangle are complementary.</td>
</tr>
<tr>
<td>4.2</td>
<td>There can be at most one right or obtuse angle in a triangle.</td>
</tr>
</tbody>
</table>

Example: \( m\angle G + m\angle J = 90 \)

You will prove Corollaries 4.1 and 4.2 in Exercises 32 and 33.

### Real-World Example: Right Angles

**Ski Jumping**  
Ski jumper Simon Ammann of Switzerland forms a right triangle with his skis and his line of sight. Find \( m\angle 2 \) if \( m\angle 1 \) is 27.

Use Corollary 4.1 to write an equation.
\[
m\angle 1 + m\angle 2 = 90
\]
\[
27 + m\angle 2 = 90 & \quad \text{Substitution} \\
m\angle 2 &= 63 & \quad \text{Subtract 27 from each side.}
\]

**Wind Surfing**  
A windsurfing sail is generally a right triangle. One of the angles that is not the right angle has a measure of 68°. What is the measure of the other nonright angle?
Find the missing angle measure.

1. 

2. 

Find each measure.

3. \( m\angle 1 \)
4. \( m\angle 2 \)
5. \( m\angle 3 \)

Find each measure in \( \triangle DEF \).

6. \( m\angle 1 \)
7. \( m\angle 2 \)

8. **Ski Jumping** American ski jumper Jessica Jerome forms a right angle with her skis. If \( m\angle 2 = 70 \), find \( m\angle 1 \).

Find the missing angle measures.

9. 

10. 

11. 

12. 

Find each measure if \( m\angle 4 = m\angle 5 \).

13. \( m\angle 1 \)
14. \( m\angle 2 \)
15. \( m\angle 3 \)
16. \( m\angle 4 \)
17. \( m\angle 5 \)
18. \( m\angle 6 \)
Lesson 4-2  Angles of Triangles

Find each measure if \( m\angle DGF = 53 \) and \( m\angle AGC = 40 \).

19. \( m\angle 1 \)
20. \( m\angle 2 \)
21. \( m\angle 3 \)
22. \( m\angle 4 \)

**SPEED SKATING** For Exercises 23–26, use the following information.

Speed skater Catriona Lemay Doan of Canada forms at least two sets of triangles and exterior angles as she skates. Use the measures of given angles to find each measure.

23. \( m\angle 1 \)
24. \( m\angle 2 \)
25. \( m\angle 3 \)
26. \( m\angle 4 \)

**HOUSING** For Exercises 27–29, use the following information.

The two braces for the roof of a house form triangles. Find each measure.

27. \( m\angle 1 \)
28. \( m\angle 2 \)
29. \( m\angle 3 \)

**PROOF** For Exercises 30–34, write the specified type of proof.

30. flow proof
   Given: \( \angle FGI \equiv \angle IGH \)
   \( \overline{GI} \perp \overline{FH} \)
   Prove: \( \angle F \equiv \angle H \)

31. two-column proof
   Given: \( ABCD \) is a quadrilateral.
   Prove: \( m\angle DAB + m\angle B + m\angle BCD + m\angle D = 360 \)

32. flow proof of Corollary 4.1
33. paragraph proof of Corollary 4.2
34. two-column proof of Theorem 4.2

**OPEN ENDDED** Draw a triangle. Label one exterior angle and its remote interior angles.

**CHALLENGE** \( \overrightarrow{BA} \) and \( \overrightarrow{BC} \) are opposite rays.

The measures of \( \angle 1, \angle 2, \) and \( \angle 3 \) are in a 4:5:6 ratio. Find the measure of each angle.
37. **FIND THE ERROR** Najee and Kara are discussing the Exterior Angle Theorem. Who is correct? Explain.

Najee: \[ m\angle 1 + m\angle 2 = m\angle 4 \]

Kara: \[ m\angle 1 + m\angle 2 + m\angle 4 = 180 \]

38. **Writing in Math** Use the information about kites provided on page 210 to explain how the angles of triangles are used to make kites. Include an explanation of how you can find the measure of a third angle if two angles of two triangles are congruent. Also include a description of the properties of two angles in a triangle if the measure of the third is 90°.

39. Two angles of a triangle have measures of 35° and 80°. Which of the following could not be a measure of an exterior angle of the triangle?

A. 165°
B. 145°
C. 115°
D. 100°

40. Which equation is equivalent to \[ 7x - 3(2 - 5x) = 8x? \]

F. \[ 2x - 6 = 8x \]
G. \[ 22x - 6 = 8x \]
H. \[ -8x - 6 = 8x \]
J. \[ 22x + 6 = 8x \]

41. Scalene
42. Obtuse
43. Isosceles

Find the distance between each pair of parallel lines. (Lesson 3-6)

44. \( y = x + 6 \), \( y = x - 10 \)
45. \( y = -2x + 3 \), \( y = -2x - 7 \)

46. **MODEL TRAINS** Regan is going to set up two parallel train tracks with a third track running diagonally across the first two. To properly place a switch, she needs the angle between the diagonal and top of the second track to be twice as large as the angle between the diagonal and top of the first track. What is the value of \( x \)? (Lesson 3-2)

47. \( \angle 1 \cong \angle 1 \) and \( \overline{AB} \cong \overline{AB} \).
48. If \( \overline{AB} \cong \overline{XY} \), then \( \overline{XY} \cong \overline{AB} \).
49. If \( \angle 1 \cong \angle 2 \), then \( \angle 2 \cong \angle 1 \).
50. If \( \angle 2 \cong \angle 3 \) and \( \angle 3 \cong \angle 4 \), then \( \angle 2 \cong \angle 4 \).
Congruent Triangles

Main Ideas

- Name and label corresponding parts of congruent triangles.
- Identify congruence transformations.

New Vocabulary

- congruent triangles
- congruence
- transformations

Corresponding Parts of Congruent Triangles

Triangles that are the same size and shape are congruent triangles. Each triangle has three angles and three sides. If all six of the corresponding parts of two triangles are congruent, then the triangles are congruent.

If \( \triangle ABC \) is congruent to \( \triangle EFG \), the vertices of the two triangles correspond in the same order as the letters naming the triangles.

\[ \triangle ABC \cong \triangle EFG \]

This correspondence of vertices can be used to name the corresponding congruent sides and angles of the two triangles.

\[ \angle A \cong \angle E \quad \angle B \cong \angle F \quad \angle C \cong \angle G \]

\[ AB \cong EF \quad BC \cong FG \quad AC \cong EG \]

The corresponding sides and angles can be determined from any congruence statement by following the order of the letters.

KEY CONCEPT

Definition of Congruent Triangles (CPCTC)

Two triangles are congruent if and only if their corresponding parts are congruent.

CPCTC stands for corresponding parts of congruent triangles are congruent. “If and only if” is used to show that both the conditional and its converse are true.
**FURNITURE DESIGN** The legs of this stool form two triangles. Suppose the measures in inches are \( QR = 12 \), \( RS = 23 \), \( QS = 24 \), \( RT = 12 \), \( TV = 24 \), and \( RV = 23 \).

a. Name the corresponding congruent angles and sides.

\[ \angle Q \cong \angle T \quad \angle QRS \cong \angle TRV \quad \angle S \cong \angle V \]

\[ QR \cong TR \quad RS \cong RV \quad QS \cong TV \]

b. Name the congruent triangles.

\[ \triangle QRS \cong \triangle TRV \]

**Check Your Progress**

The measures of the sides of triangles \( PDQ \) and \( OEC \) are \( PD = 5 \), \( DQ = 7 \), \( PQ = 11 \); \( EC = 7 \), \( OC = 5 \), and \( OE = 11 \).

1A. Name the corresponding congruent angles and sides.

1B. Name the congruent triangles.

Like congruence of segments and angles, congruence of triangles is reflexive, symmetric, and transitive.

**Theorem 4.4**

**Properties of Triangle Congruence**

Congruence of triangles is reflexive, symmetric, and transitive.

<table>
<thead>
<tr>
<th>Reflexive</th>
<th>Transitive</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \triangle JKL \cong \triangle JKL )</td>
<td>If ( \triangle JKL \cong \triangle PQR ), and ( \triangle PQR \cong \triangle XYZ ), then ( \triangle JKL \cong \triangle XYZ ).</td>
</tr>
<tr>
<td>If ( \triangle JKL \cong \triangle PQR ), then ( \triangle PQR \cong \triangle JKL ).</td>
<td></td>
</tr>
</tbody>
</table>

You will prove the symmetric and reflexive parts of Theorem 4.4 in Exercises 30 and 32, respectively.

**Proof**

**Theorem 4.4 (Transitive)**

**Given:** \( \triangle ABC \cong \triangle DEF \)

\( \triangle DEF \cong \triangle GHI \)

**Prove:** \( \triangle ABC \cong \triangle GHI \)

**Proof:** You are given that \( \triangle ABC \cong \triangle DEF \). Because corresponding parts of congruent triangles are congruent, \( \angle A \cong \angle D \), \( \angle B \cong \angle E \), \( \angle C \cong \angle F \), \( AB \cong DE \), \( BC \cong EF \), and \( AC \cong DF \). You are also given that \( \triangle DEF \cong \triangle GHI \). So \( \angle D \cong \angle G \), \( \angle E \cong \angle H \), \( \angle F \cong \angle I \), \( DE \cong GH \), \( EF \cong HI \), and \( DF \cong GI \), by CPCTC. Therefore, \( \angle A \cong \angle G \), \( \angle B \cong \angle H \), \( \angle C \cong \angle I \), \( AB \cong GH \), \( BC \cong HI \), and \( AC \cong GI \) because congruence of angles and segments is transitive. Thus, \( \triangle ABC \cong \triangle GHI \) by the definition of congruent triangles.
Identify Congruence Transformations  In the figures below, \( \triangle ABC \) is congruent to \( \triangle DEF \). If you slide, or translate, \( \triangle DEF \) up and to the right, \( \triangle DEF \) is still congruent to \( \triangle ABC \).

The congruency does not change whether you turn, or rotate, \( \triangle DEF \) or flip, or reflect, \( \triangle DEF \). \( \triangle ABC \) is still congruent to \( \triangle DEF \).

If you slide, flip, or turn a triangle, the size and shape do not change. These three transformations are called congruence transformations.

**Example**  Transformations in the Coordinate Plane

**COORDINATE GEOMETRY**  The vertices of \( \triangle CDE \) are \( C(-5, 7), D(-8, 6), \) and \( E(-3, 3) \). The vertices of \( \triangle C'D'E' \) are \( C'(5, 7), D'(8, 6), \) and \( E'(3, 3) \).

**a. Verify that \( \triangle CDE \cong \triangle C'D'E' \).**

Use the Distance Formula to find the length of each side in the triangles.

\[
\begin{align*}
DC &= \sqrt{(-8 - (-5))^2 + (6 - 7)^2} \\
&= \sqrt{9 + 1} \text{ or } \sqrt{10}
\end{align*}
\]

\[
\begin{align*}
D' C' &= \sqrt{(8 - 5)^2 + (6 - 7)^2} \\
&= \sqrt{9 + 1} \text{ or } \sqrt{10}
\end{align*}
\]

\[
\begin{align*}
DE &= \sqrt{(-8 - (-3))^2 + (6 - 3)^2} \\
&= \sqrt{25 + 9} \text{ or } \sqrt{34}
\end{align*}
\]

\[
\begin{align*}
D' E' &= \sqrt{(8 - 3)^2 + (6 - 3)^2} \\
&= \sqrt{25 + 9} \text{ or } \sqrt{34}
\end{align*}
\]

\[
\begin{align*}
CE &= \sqrt{(-5 - (-3))^2 + (7 - 3)^2} \\
&= \sqrt{4 + 16} \\
&= 2\sqrt{5}
\end{align*}
\]

\[
\begin{align*}
C' E' &= \sqrt{(5 - 3)^2 + (7 - 3)^2} \\
&= \sqrt{4 + 16} \\
&= 2\sqrt{5}
\end{align*}
\]

By the definition of congruence, \( DC \cong D'C', DE \cong D'E', \) and \( CE \cong C'E' \).

Use a protractor to measure the angles of the triangles. You will find that the measures are the same.

In conclusion, because \( DC \cong D'C', DE \cong D'E', \) and \( CE \cong C'E', \angle D \cong \angle D', \angle C \cong \angle C', \) and \( \angle E \cong \angle E', \triangle CDE \cong \triangle C'D'E'.

(continued on the next page)
b. Name the congruence transformation for \( \triangle CDE \) and \( \triangle C'D'E' \).
\( \triangle C'D'E' \) is a flip, or reflection, of \( \triangle CDE \).

**COORDINATE GEOMETRY**
The vertices of \( \triangle LMN \) are \( L(1, 1) \), \( M(3, 5) \),
and \( N(5, 1) \). The vertices of \( \triangle L'M'N' \) are \( L'(-1, -1) \), \( M'(-3, -5) \), and \( N'(-5, -1) \).

2A. Verify that \( \triangle LMN \cong \triangle L'M'N' \).
2B. Name the congruence transformation for \( \triangle LMN \) and \( \triangle L'M'N' \).

**Example 1**
(p. 218)
Identify the corresponding congruent angles and sides and the congruent triangles in each figure.

1. 

2. 

3. **QUILTING**
   In the quilt design, assume that angles and segments that appear to be congruent are congruent. Indicate which triangles are congruent.

4. The vertices of \( \triangle SL'UV' \) and \( \triangle S'U'V' \) are \( S(0, 4), U(0, 0), V(2, 2), S'(0, -4), U'(0, 0), \) and \( V'(-2, -2) \). Verify that the triangles are congruent and then name the congruence transformation.

5. The vertices of \( \triangle QRT \) and \( \triangle Q'R'T' \) are \( Q(-4, 3), Q'(4, 3), R(-4, -2), R'(4, -2), T(-1, -2), \) and \( T'(1, -2) \). Verify that \( \triangle QRT \cong \triangle Q'R'T' \). Then name the congruence transformation.

**Example 2**
(p. 219)

**Exercises**
Identify the congruent angles and sides and the congruent triangles in each figure.

6. 

7. 

---

220  Chapter 4  Congruent Triangles
Identify the congruent angles and sides and the congruent triangles in each figure.

8. [Diagram of ΔPQV and ΔP'Q'V']

Verify each congruence and name the congruence transformation.

10. ΔPQV ≅ ΔP'Q'V'

11. ΔMNP ≅ ΔM'N'P'

12. ΔGHF ≅ ΔG'H'F'

13. ΔJKL ≅ ΔJ'K'L'

Name the congruent angles and sides for each pair of congruent triangles.

14. ΔTUV ≅ ΔXYZ
15. ΔCDG ≅ ΔRSW
16. ΔBCF ≅ ΔDGH
17. ΔADG ≅ ΔHKL

18. UMBRELLAS Umbrellas usually have eight triangular sections with ribs of equal length. Are the statements ΔJAD ≅ ΔIAE and ΔJAD ≅ ΔEAI both correct? Explain.

19. MOSAICS The figure at the left is the center of a Roman mosaic. If the bases of the triangles are each the same length, what else do you need to know to conclude that the four triangles surrounding the square are congruent?

Assume that segments and angles that appear to be congruent in each figure are congruent. Indicate which triangles are congruent.

20. [Diagram of a figure with labeled triangles]
21. [Diagram of a figure with labeled triangles]
22. [Diagram of a figure with labeled triangles]

Real-World Link

A mosaic is composed of glass, marble, or ceramic pieces often arranged in a pattern. The pieces, or tesserae, are set in cement. Mosaics are used to decorate walls, floors, and gardens.

Source: www.dimosaic.com
Determine whether each statement is true or false. Draw an example or counterexample for each.

23. Two triangles with corresponding congruent angles are congruent.

24. Two triangles with angles and sides congruent are congruent.

**ALGEBRA** For Exercises 25 and 26, use the following information.
\( \triangle QRS \cong \triangle GHJ \), \( RS = 12 \), \( QR = 10 \), \( QS = 6 \), and \( HJ = 2x - 4 \).

25. Draw and label a figure to show the congruent triangles.

26. Find \( x \).

**ALGEBRA** For Exercises 27 and 28, use the following information.
\( \triangle JKL \cong \triangle DEF \), \( m\angle J = 36 \), \( m\angle E = 64 \), and \( m\angle F = 3x + 52 \).

27. Draw and label a figure to show the congruent triangles.

28. Find \( x \).

29. **GARDENING** This garden lattice will be covered with morning glories in the summer. Malina wants to save two triangular areas for artwork. If \( \triangle GHJ \cong \triangle KLP \), name the corresponding congruent angles and sides.

30. **PROOF** Put the statements used to prove the statement below in the correct order. Provide the reasons for each statement.

   **Congruence of triangles is symmetric.**

   Given: \( \triangle RST \cong \triangle XYZ \)
   
   Prove: \( \triangle XYZ \cong \triangle RST \)
   
   Proof:
   
   - \( \angle X \equiv \angle R, \angle Y \equiv \angle L, \angle Z \equiv \angle T \)
   - \( RS \equiv XY, ST \equiv YZ, RT \equiv XZ \)
   - \( \triangle RST \equiv \triangle XYZ \)
   - \( \triangle XYZ \equiv \triangle RST \)

31. **PROOF** Copy the flow proof and provide the reasons for each statement.

   Given: \( AB \parallel CD, AD \parallel BC \), \( AB \perp BC \), \( AB \parallel CD \), \( AD \parallel DC \), \( AB \perp BC \), \( AD \parallel DC \)
   
   Prove: \( \triangle ACD \cong \triangle CAB \)
   
   Proof:
   
   - \( AB \parallel CD \)
   - \( AD \parallel CB \)
   - \( AC \parallel CA \)
   - \( AD \perp DC \)
   - \( AB \perp BC \)
   - \( AD \parallel BC \)
   - \( AB \parallel CD \)
   - \( \angle D \) is a rt. \( \angle \)
   - \( \angle B \) is a rt. \( \angle \)
   - \( \angle 1 \equiv \angle 4 \)
   - \( \angle 2 \equiv \angle 3 \)
   - \( \angle D \equiv \angle B \)
   - \( \angle D \) is a rt. \( \angle \)
   - \( \angle B \) is a rt. \( \angle \)
   - \( \angle 1 \equiv \angle 4 \)
   - \( \angle 2 \equiv \angle 3 \)
   - \( \Delta ACD \cong \Delta CAB \)

**EXTRA PRACTICE**
See pages 807, 831.

Self-Check Quiz at geometryonline.com
32. **PROOF** Write a flow proof to prove that congruence of triangles is reflexive. *(Theorem 4.4)*

33. **H.O.T. Problems** Find a real-world picture of congruent triangles and explain how you know that the triangles are congruent.

34. **CHALLENGE** \(\triangle RST\) is isosceles with \(RS = RT\), \(M\), \(N\), and \(P\) are midpoints of the respective sides, \(\angle S \cong \angle MPS\), and \(NP \cong MP\). What else do you need to know to prove that \(\triangle SMP \cong \triangle TNP\)?

35. **Writing in Math** Use the information on page 217 to explain why triangles are used in the design and construction of bridges.

36. Triangle \(ABC\) is congruent to \(\triangle HIJ\). The vertices of \(\triangle ABC\) are \(A(-1, 2)\), \(B(0, 3)\), and \(C(2, -2)\). What is the measure of side \(HJ\)?
   - A \(\sqrt{2}\)
   - B 3
   - C 5
   - D cannot be determined

37. **REVIEW** Which is a factor of \(x^2 + 19x - 42\)?
   - F \(x + 14\)
   - G \(x + 2\)
   - H \(x - 14\)
   - J \(x - 2\)

38. Bryssa cut four congruent triangles off the corners of a rectangle to make an octagon as shown below.

   What is the area of the octagon?
   - A 456 cm\(^2\)
   - B 528 cm\(^2\)
   - C 552 cm\(^2\)
   - D 564 cm\(^2\)

39. Find \(x\). *(Lesson 4-2)*

40. Find \(x\) and the measure of each side of the triangle. *(Lesson 4-1)*

41. \(\triangle BCD\) is isosceles with \(BC \cong CD\), \(BC = 2x + 4\), \(BD = x + 2\) and \(CD = 10\).

42. \(\triangle HKT\) is equilateral with \(HK = x + 7\) and \(HT = 4x - 8\).

43. **PREREQUISITE SKILL** Find the distance between each pair of points. *(Lesson 1-3)*

44. \((-1, 7), (1, 6)\)

45. \((8, 2), (4, -2)\)

46. \((3, 5), (5, 2)\)

47. \((0, -6), (-3, -1)\)
Making Concept Maps

When studying a chapter, it is wise to record the main topics and vocabulary you encounter. In this chapter, some of the new vocabulary words were triangle, acute triangle, obtuse triangle, right triangle, equiangular triangle, scalene triangle, isosceles triangle, and equilateral triangle. The triangles are all related by the size of the angles or the number of congruent sides.

A graphic organizer called a concept map is a convenient way to show these relationships. A concept map is shown below for the different types of triangles. The main ideas are in boxes. Any information that describes how to move from one box to the next is placed along the arrows.

Reading to Learn

1. Describe how to use the concept map to classify triangles by their side lengths.

2. In \( \triangle ABC \), \( m\angle A = 48 \), \( m\angle B = 41 \), and \( m\angle C = 91 \). Use the concept map to classify \( \triangle ABC \).

3. Identify the type of triangle that is linked to both classifications.
Around 120 B.C., Greek developers and land owners used the properties of geometry to accurately and precisely divide plots of land. Since that time, surveying has been used in areas such as map making and engineering. To check a measurement, land surveyors mark out a right triangle and then mark a second triangle that is congruent to the first.

**SSS Postulate** Is it always necessary to show that all of the corresponding parts of two triangles are congruent to prove that the triangles are congruent? In this lesson, we will explore two other methods to prove that triangles are congruent.

Use the following construction to construct a triangle with sides that are congruent to a given \( \triangle XYZ \).

**CONSTRUCTION**

**Congruent Triangles Using Sides**

*Step 1* Use a straightedge to draw any line \( \ell \), and select a point \( R \). Use a compass to construct \( RS \) on \( \ell \), such that \( RS = XZ \).

*Step 2* Using \( R \) as the center, draw an arc with radius equal to \( XY \).

*Step 3* Using \( S \) as the center, draw an arc with radius equal to \( YZ \).

*Step 4* Let \( T \) be the point of intersection of the two arcs. Draw \( RT \) and \( ST \) to form \( \triangle RST \).

*Step 5* Cut out \( \triangle RST \) and place it over \( \triangle XYZ \). How does \( \triangle RST \) compare to \( \triangle XYZ \)?
If the corresponding sides of two triangles are congruent, then the triangles are congruent. This is the Side-Side-Side Postulate and is written as SSS.

### POSTULATE 4.1 Side-Side-Side Congruence

If the sides of one triangle are congruent to the sides of a second triangle, then the triangles are congruent.

**Abbreviation:** SSS

![Diagram](Image)

**MARINE BIOLOGY** The tail of an orca whale can be viewed as two triangles that share a common side. Write a two-column proof to prove that \( \triangle BXA \cong \triangle CXA \) if \( AB \cong AC \) and \( BX \cong CX \).

**Given:** \( AB \cong AC; BX \cong CX \)

**Prove:** \( \triangle BXA \cong \triangle CXA \)

**Proof:**

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( AB \cong AC; BX \cong CX )</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. ( AX \cong AX )</td>
<td>2. Reflexive Property</td>
</tr>
<tr>
<td>3. ( \triangle BXA \cong \triangle CXA )</td>
<td>3. SSS</td>
</tr>
</tbody>
</table>

**1A.** A “Caution, Floor Slippery When Wet” sign is composed of three triangles. If \( AB \cong AD \) and \( CB \cong DC \), prove that \( \triangle ACB \cong \triangle ACD \).

**1B.** Triangle \( QRS \) is an isosceles triangle with \( QR \cong RS \). If there exists a line \( RT \) that bisects \( \angle QRS \) and \( QS \), show that \( \triangle QRT \cong \triangle SRT \).
You can use the Distance Formula and postulates about triangle congruence to relate figures on the coordinate plane.

**Example SSS on the Coordinate Plane**

**COORDINATE GEOMETRY** Determine whether \( \triangle RTZ \cong \triangle JKL \) for \( R(2, 5) \), \( Z(1, 1) \), \( T(5, 2) \), \( L(-3, 0) \), \( K(-7, 1) \), and \( J(-4, 4) \). Explain.

Use the Distance Formula to show that the corresponding sides are congruent.

\[
RT = \sqrt{(2 - 5)^2 + (5 - 2)^2} = \sqrt{9 + 9} = \sqrt{18} \text{ or } 3\sqrt{2}
\]

\[
JK = \sqrt{[-4 - (-7)]^2 + (4 - 1)^2} = \sqrt{9 + 9} = \sqrt{18} \text{ or } 3\sqrt{2}
\]

\[
TZ = \sqrt{(5 - 1)^2 + (2 - 1)^2} = \sqrt{16 + 1} \text{ or } \sqrt{17}
\]

\[
KL = \sqrt{[-7 - (-3)]^2 + (1 - 0)^2} = \sqrt{16 + 1} \text{ or } \sqrt{17}
\]

\[
RZ = \sqrt{(2 - 1)^2 + (5 - 1)^2} = \sqrt{1 + 16} \text{ or } \sqrt{17}
\]

\[
JL = \sqrt{[-4 - (-3)]^2 + (4 - 0)^2} = \sqrt{1 + 16} \text{ or } \sqrt{17}
\]

\( RT = JK \), \( TZ = KL \), and \( RZ = JL \). By definition of congruent segments, all corresponding segments are congruent. Therefore, \( \triangle RTZ \cong \triangle JKL \) by SSS.

**Check Your Progress**

2. Determine whether triangles \( \triangle ABC \) and \( \triangle TDS \) with vertices \( A(1, 1) \), \( B(3, 2) \), \( C(2, 5) \), \( T(1, -1) \), \( D(3, -3) \), and \( S(2, -5) \) are congruent. Justify your reasoning.

**SAS Postulate** Suppose you are given the measures of two sides and the angle they form, called the **included angle**. These conditions describe a unique triangle. Two triangles in which corresponding sides and the included pairs of angles are congruent provide another way to show that triangles are congruent.

**Postulate 4.2 Side-Angle-Side Congruence**

If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, then the triangles are congruent.

**Abbreviation: SAS**

\( \triangle ABC \cong \triangle FDE \)
You can also construct congruent triangles given two sides and the included angle.

**CONSTRUCTION**

**Congruent Triangles Using Two Sides and the Included Angle**

**Step 1** Draw a triangle and label its vertices $A$, $B$, and $C$.

**Step 2** Select a point $K$ on line $m$. Use a compass to construct $KL$ on $m$ such that $KL \cong BC$.

**Step 3** Construct an angle congruent to $\angle B$ using $KL$ as a side of the angle and point $K$ as the vertex.

**Step 4** Construct $\overline{JK}$ such that $JK \cong AB$. Draw $\overline{JL}$ to complete $\triangle JKL$.

**Step 5** Cut out $\triangle JKL$ and place it over $\triangle ABC$. How does $\triangle JKL$ compare to $\triangle ABC$?

**EXAMPLE** Use SAS in Proofs

Write a flow proof.

**Given:** $X$ is the midpoint of $\overline{BD}$. $X$ is the midpoint of $\overline{AC}$.

**Prove:** $\triangle DXC \cong \triangle BXA$

**Flow Proof:**

- **Given**
  - $X$ is the midpoint of $\overline{BD}$
  - $X$ is the midpoint of $\overline{AC}$

- **Midpoint Theorem**
  - $\overline{DX} \cong \overline{BX}$
  - $\overline{CX} \cong \overline{AX}$

- **SAS**
  - $\triangle DXC \cong \triangle BXA$

**Vertical $\angle$ are $\equiv$.**

**CHECK Your Progress**

3. The spokes used in a captain’s wheel divide the wheel into eight parts. If $\overline{TU} \cong \overline{TX}$ and $\angle XTV \cong \angle UTV$, show that $\triangle XTV \cong \triangle UTV$.

**Personal Tutor at geometryonline.com**
EXAMPLE 4 Identify Congruent Triangles

Determine which postulate can be used to prove that the triangles are congruent. If it is not possible to prove that they are congruent, write not possible.

a.

Each pair of corresponding sides are congruent. The triangles are congruent by the SSS Postulate.

b.

The triangles have three pairs of corresponding angles congruent. This does not match the SSS or the SAS Postulate. It is not possible to prove them congruent.

CHECK Your Progress

4A.

4B.

Example 1 (p. 226)

1. JETS The United States Navy Flight Demonstration Squadron, the Blue Angels, fly in a formation that can be viewed as two triangles with a common side. Write a two-column proof to prove that \( \triangle SRT \cong \triangle QRT \) if \( T \) is the midpoint of \( SQ \) and \( SR \cong QR \).

Example 2 (p. 227)

Determine whether \( \triangle EFG \cong \triangle MNP \) given the coordinates of the vertices. Explain.

2. \( E(-4, -3), F(-2, 1), G(-2, -3), M(4, -3), N(2, 1), P(2, -3) \)

3. \( E(-2, -2), F(-4, 6), G(-3, 1), M(2, 2), N(4, 6), P(3, 1) \)

Example 3 (p. 228)

4. CATS A cat’s ear is triangular in shape. Write a proof to prove \( \triangle RST \cong \triangle PNM \) if \( RS \cong PN, RT \cong PM, \angle S \cong \angle N, \) and \( \angle T \cong \angle M \).

Example 4 (p. 229)

Determine which postulate can be used to prove that the triangles are congruent. If it is not possible to prove that they are congruent, write not possible.

5.

6.
PROOF For Exercises 7 and 8, write a two-column proof.

7. Given: ΔCDE is an isosceles triangle. G is the midpoint of CE.
Prove: ΔCDG ≅ ΔEDG

8. Given: \overline{AC} \cong \overline{GC} \overline{EC} \text{ bisects } \overline{AG}.
Prove: ΔGEC ≅ ΔAEC

Determine whether ΔJKL ≅ ΔFGH given the coordinates of the vertices. Explain.

9. J(2, 5), K(5, 2), L(1, 1), F(−4, 4), G(−7, 1), H(−3, 0)
10. J(−1, 1), K(−2, −2), L(−5, −1), F(2, −1), G(3, −2), H(2, 5)
11. J(−1, −1), K(0, 6), L(2, 3), F(3, 1), G(5, 3), H(8, 1)
12. J(3, 9), K(4, 6), L(1, 5), F(1, 7), G(2, 4), H(−1, 3)

PROOF For Exercises 13 and 14, write the specified type of proof.

13. flow proof
Given: \overline{KM} \parallel \overline{LJ}, \overline{KM} \cong \overline{LJ}
Prove: ΔJKM ≅ ΔMLJ

14. two-column proof
Given: \overline{DE} and \overline{BC} bisect each other.
Prove: ΔDGB ≅ ΔEGC

Determine which postulate can be used to prove that the triangles are congruent. If it is not possible to prove that they are congruent, write not possible.

15. 16. 17. 18.

PROOF For Exercises 19 and 20, write a flow proof.

19. Given: \overline{AE} \cong \overline{CF}, \overline{AB} \cong \overline{CB}, \overline{BE} \cong \overline{BF}
Prove: ΔAFB ≅ ΔCEB

20. Given: \overline{RQ} \cong \overline{TQ} \cong \overline{YQ} \cong \overline{WQ}
∠RQY ≅ ∠WQT
Prove: ΔQWT ≅ ΔQYR
21. **GESE** A flock of geese flies in formation. Write a proof to prove that \( \triangle EFG \cong \triangle HFG \) if \( EF \cong HF \) and \( G \) is the midpoint of \( EH \).

**PROOF** For Exercises 22 and 23, write a two-column proof.

22. **Given:** \( \triangle MRN \cong \triangle QRP \)
\( \angle MNP \cong \angle QPN \)
**Prove:** \( \triangle MNP \cong \triangle QPN \)

23. **Given:** \( \triangle GHJ \cong \triangle LKJ \)
**Prove:** \( \triangle GHL \cong \triangle LKG \)

**BASEBALL** For Exercises 24 and 25, use the following information.
A baseball diamond is a square with four right angles and all sides congruent.

24. Write a two-column proof to prove that the distance from first base to third base is the same as the distance from home plate to second base.

25. Write a two-column proof to prove that the angle formed between second base, home plate, and third base is the same as the angle formed between second base, home plate, and first base.

26. **REASONING** Explain how the SSS postulate can be used to prove that two triangles are congruent.

27. **OPEN ENDED** Find two triangles in a newspaper or magazine and show that they are congruent.

28. **FIND THE ERROR** Carmelita and Jonathan are trying to determine whether \( \triangle ABC \) is congruent to \( \triangle DEF \). Who is correct and why?

Carmelita
\( \triangle ABC \cong \triangle DEF \)
**by SAS**

Jonathan
Congruence cannot be determined.

29. **CHALLENGE** Devise a plan and write a two-column proof for the following.
**Given:** \( DE \cong FB, AE \cong FC \),
\( AE \perp DB, CF \perp DB \)
**Prove:** \( \triangle ABD \cong \triangle CDB \)

30. **Writing in Math** Describe two different methods that could be used to prove that two triangles are congruent.
31. Which of the following statements about the figure is true?

A  \( a + b < 90 \)  
B  \( a + b > 90 \)  
C  \( a + b = 90 \)  
D  \( a + b = 45 \)

32. REVIEW The Murphy family just drove 300 miles to visit their grandparents. Mr. Murphy drove 70 mph for 65% of the trip and 35 mph or less for 20% of the trip that was left. Assuming that Mr. Murphy never went over 70 mph, how many miles did he travel at a speed between 35 and 70 mph?

F 195  
H 21  
G 84  
J 18

33. Identify the congruent triangles in each figure. (Lesson 4-3)

34.

35.

Find each measure if \( \overline{PQ} \perp \overline{QR} \). (Lesson 4-2)

36. \( m\angle 2 \)  
37. \( m\angle 3 \)  
38. \( m\angle 5 \)  
39. \( m\angle 4 \)  
40. \( m\angle 1 \)  
41. \( m\angle 6 \)

ANALYZE GRAPHS For Exercises 42 and 43, use the graph of sales of a certain video game system in a recent year. (Lesson 3-3)

42. Find the rate of change from first quarter to the second quarter.

43. Which had the greater rate of change: first quarter to second quarter, or third to fourth?

PREREQUISITE SKILL \( \overline{BD} \) and \( \overline{AE} \) are angle bisectors and segment bisectors. Name the indicated segments and angles. (Lessons 1-5 and 1-6)

44. segment congruent to \( \overline{EC} \)  
45. angle congruent to \( \angle ABD \)  
46. angle congruent to \( \angle BDC \)  
47. segment congruent to \( \overline{AD} \)  
48. angle congruent to \( \angle BAE \)  
49. angle congruent to \( \angle BXA \)
1. **MULTIPLE CHOICE** Classify \( \triangle ABC \) with vertices \( A(-1, 1), B(1, 3), \) and \( C(3, -1) \). (Lesson 4-1)

A scalene acute  
B equilateral  
C isosceles acute  
D isosceles right

2. Identify the isosceles triangles in the figure, if \( FH \) and \( DG \) are congruent perpendicular bisectors. (Lesson 4-1)


![Diagram of \( \triangle ABC \)]

\( \triangle ABC \) is equilateral with \( AB = 2x, \) 
\( BC = 4x - 7, \) and \( AC = x + 3.5 \). (Lesson 4-1)

3. Find \( x \).

4. Find the measure of each side.

Find the measure of each angle listed below. (Lesson 4-2)

5. \( m\angle 1 \)

6. \( m\angle 2 \)

7. \( m\angle 3 \)

Find each measure. (Lesson 4-2)

8. \( m\angle 1 \)

9. \( m\angle 2 \)

10. \( m\angle 3 \)

11. Find the missing angle measures. (Lesson 4-2)

12. If \( \triangle MNP \equiv \triangle JKL \), name the corresponding congruent angles and sides. (Lesson 4-3)

13. **MULTIPLE CHOICE** Given: \( \triangle ABC \equiv \triangle XYZ \). Which of the following must be true? (Lesson 4-3)

F \( \angle A \equiv \angle Y \)

G \( \overline{AC} \equiv \overline{XZ} \)

H \( \overline{AB} \equiv \overline{YZ} \)

J \( \angle Z \equiv \angle B \)

**COORDINATE GEOMETRY** The vertices of \( \triangle JKL \) are \( J(7, 7), K(3, 7), L(7, 1) \). The vertices of \( \triangle J'K'L' \) are \( J'(-7, -7), K'(-3, -7), L'(-7, -1) \). (Lesson 4-3)

14. Verify that \( \triangle JKL \equiv \triangle J'K'L' \).

15. Name the congruence transformation for \( \triangle JKL \) and \( \triangle J'K'L' \).

16. Determine whether \( \triangle JML \equiv \triangle BDG \) given that \( J(-4, 5), M(-2, 6), L(-1, 1), B(-3, -4), D(-4, -2), \) and \( G(1, -1) \). (Lesson 4-4)

17. Determine whether \( \triangle XYZ \equiv \triangle TUV \) given the coordinates of the vertices. Explain. (Lesson 4-4)

18. \( X(0, 0), Y(3, 3), Z(0, 3), T(-6, -6), U(-3, -3), V(-3, -6) \)

19. \( X(7, 0), Y(5, 4), Z(1, 1), T(-5, -4), U(-3, 4), V(1,1) \)

20. Given: \( \triangle ABF \equiv \triangle EDF \)

\( \overline{CF} \) is angle bisector of \( \angle DFB \).

Prove: \( \triangle BCF \equiv \triangle DCF \).

Chapter 4 Mid-Chapter Quiz
The Bank of China Tower in Hong Kong has triangular trusses for structural support. These trusses form congruent triangles. In this lesson, we will explore two additional methods of proving triangles congruent.

**ASA Postulate** Suppose you were given the measures of two angles of a triangle and the side between them, the included side. Do these measures form a unique triangle?

**CONSTRUCTION**

**Congruent Triangles Using Two Angles and Included Side**

1. **Step 1** Draw a triangle and label its vertices \(A\), \(B\), and \(C\).
2. **Step 2** Draw any line \(m\) and select a point \(L\). Construct \(LK\) such that \(LK \cong CB\).
3. **Step 3** Construct an angle congruent to \(\angle C\) at \(L\) using \(LK\) as a side of the angle.
4. **Step 4** Construct an angle congruent to \(\angle B\) at \(K\) using \(LK\) as a side of the angle. Label the point where the new sides of the angles meet \(J\).
5. **Step 5** Cut out \(\triangle JKL\) and place it over \(\triangle ABC\). How does \(\triangle JKL\) compare to \(\triangle ABC\)?

This construction leads to the Angle-Side-Angle Postulate, written as ASA.
**POSTULATE 4.3**

Angle-Side-Angle Congruence

If two angles and the included side of one triangle are congruent to two angles and the included side of another triangle, then the triangles are congruent.

Abbreviation: ASA

![Diagram of triangles RTW and CGH with angles and sides labeled]

**EXAMPLE** Use ASA in Proofs

Write a paragraph proof.

Given: \( \overline{CP} \) bisects \( \angle BCR \) and \( \angle BPR \).

Prove: \( \triangle BCP \cong \triangle RCP \)

**Proof:** Since \( \overline{CP} \) bisects \( \angle BCR \) and \( \angle BPR \), \( \angle BCP \cong \angle RPC \) and \( \angle BPC \cong \angle RPC \). \( \overline{CP} \cong \overline{CP} \) by the Reflexive Property. By ASA, \( \triangle BCP \cong \triangle RCP \).

**CHECK Your Progress**

1. **Given:** \( \angle CAD \cong \angle BDA \) and \( \angle CDA \cong \angle BAD \)
   **Prove:** \( \triangle ABD \cong \triangle DCA \)

**AAS Theorem** Suppose you are given the measures of two angles and a nonincluded side. Is this information sufficient to prove two triangles congruent?

---

**GEOMETRY LAB**

### Angle-Angle-Side Congruence

**MODEL**

**Step 1** Draw a triangle on a piece of patty paper. Label the vertices \( A, B, \) and \( C \).

![Triangle labeled ABC]

**Step 2** Copy \( \overline{AB} \), \( \angle B \), and \( \angle C \) on another piece of patty paper and cut them out.

![Copied triangle ABC]

**Step 3** Assemble them to form a triangle in which the side is not the included side of the angles.

![Assembled triangle]

**ANALYZE**

1. Place the original \( \triangle ABC \) over the assembled figure. How do the two triangles compare?
2. **Make a conjecture** about two triangles with two angles and the nonincluded side of one triangle congruent to two angles and the nonincluded side of the other triangle.
This lab leads to the Angle-Angle-Side Theorem, written as AAS.

**THEOREM 4.5 Angle-Angle-Side Congruence**

If two angles and a nonincluded side of one triangle are congruent to the corresponding two angles and side of a second triangle, then the two triangles are congruent.

Abbreviation: AAS

**Example:** \( \triangle JKL \cong \triangle CAB \)

**PROOF Theorem 4.5**

**Given:** \( \angle M \cong \angle S, \angle J \cong \angle R, \overline{MP} \cong \overline{ST} \)

**Prove:** \( \triangle JMP \cong \triangle RST \)

**Proof:**

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \angle M \cong \angle S, \angle J \cong \angle R, \overline{MP} \cong \overline{ST} )</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. ( \angle P \cong \angle T )</td>
<td>2. Third Angle Theorem</td>
</tr>
<tr>
<td>3. ( \triangle JMP \cong \triangle RST )</td>
<td>3. ASA</td>
</tr>
</tbody>
</table>

**EXAMPLE Use AAS in Proofs**

Write a flow proof.

**Given:** \( \angle EAD \cong \angle EBC \)

\( AD \cong BC \)

**Prove:** \( AE \cong BE \)

**Flow Proof:**

1. \( \angle EAD \cong \angle EBC \)
   - Given
2. \( AD \cong BC \)
   - Given
3. \( \triangle ADE \cong \triangle BCE \)
   - AAS
4. \( AE \cong BE \)
   - CPCTC
5. \( \angle LE \cong \angle LE \)
   - Reflexive Property

**CHECK Your Progress**

2. Write a flow proof.

**Given:** \( \overline{RQ} \cong \overline{ST} \) and \( \overline{RQ} \parallel \overline{ST} \)

**Prove:** \( \triangle RUQ \cong \triangle TUS \)

You have learned several methods for proving triangle congruence. The Concept Summary lists ways to help you determine which method to use.
**Method Summary**

<table>
<thead>
<tr>
<th>Method</th>
<th>Use when...</th>
</tr>
</thead>
<tbody>
<tr>
<td>Definition of Congruent Triangles</td>
<td>All corresponding parts of one triangle are congruent to the corresponding parts of the other triangle.</td>
</tr>
<tr>
<td>SSS</td>
<td>The three sides of one triangle are congruent to the three sides of the other triangle.</td>
</tr>
<tr>
<td>SAS</td>
<td>Two sides and the included angle of one triangle are congruent to two sides and the included angle of the other triangle.</td>
</tr>
<tr>
<td>ASA</td>
<td>Two angles and the included side of one triangle are congruent to two angles and the included side of the other triangle.</td>
</tr>
<tr>
<td>AAS</td>
<td>Two angles and a nonincluded side of one triangle are congruent to two angles and side of the other triangle.</td>
</tr>
</tbody>
</table>

**Real-World EXAMPLE**

**ARCHITECTURE** This glass chapel was designed by Frank Lloyd Wright’s son, Lloyd Wright. Suppose the redwood supports, $\overline{TU}$ and $\overline{TV}$, measure 3 feet, $TY = 1.6$ feet, and $m\angle U$ and $m\angle V$ are 31. Determine whether $\triangle TYU \cong \triangle TYV$. Justify your answer.

**Explore** We are given three measurements of each triangle. We need to determine whether the two triangles are congruent.

**Plan** Since $m\angle U = m\angle V$, $\angle U \cong \angle V$. Likewise, $TU = TV$ so $\overline{TU} \cong \overline{TV}$, and $TY = TY$ so $\overline{TY} \cong \overline{TY}$. Check each possibility using the five methods you know.

**Solve** We are given information about side-side-angle (SSA). This is not a method to prove two triangles congruent.

**Check** Use a compass, protractor, and ruler to draw a triangle with the given measurements. For space purposes, use centimeters instead of feet.

- Draw a segment 3.0 centimeters long.
- At one end, draw an angle of 31°. Extend the line longer than 3.0 centimeters.
- At the other end, draw an arc with a radius of 1.6 centimeters such that it intersects the line.

Notice that there are two possible segments that could determine the triangle. Since the given measurements do not lead to a unique triangle, we cannot show that the triangles are congruent.

(continued on the next page)

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Extra Examples at geometryonline.com

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For Exercises 1–4, write the specified type of proof.

1. flow proof  
   **Given:** $GH \parallel KJ, \overline{GK} \parallel HJ$  
   **Prove:** $\triangle GJK \cong \triangle JGH$  

2. paragraph proof  
   **Given:** $\angle E \cong \angle K, \angle DGH \cong \angle DHG, \overline{EG} \cong \overline{KH}$  
   **Prove:** $\triangle EGD \cong \triangle KHD$  

3. paragraph proof  
   **Given:** $\overline{QS}$ bisects $\angle RST; \angle R \cong \angle T$  
   **Prove:** $\triangle QRS \cong \triangle QTS$  

4. flow proof  
   **Given:** $\overline{XW} \parallel \overline{YZ}, \angle X \cong \angle Z$  
   **Prove:** $\triangle WXY \cong \triangle YZW$  

5. **PARACHUTES** Suppose $\overline{ST}$ and $\overline{ML}$ each measure seven feet, $\overline{SR}$ and $\overline{MK}$ each measure 5.5 feet, and $m\angle T = m\angle L = 49$. Determine whether $\triangle SRT \cong \triangle MKL$. Justify your answer.
Write a paragraph proof.
6. Given: $\angle NOM \cong \angle POR$, $\overline{NM} \perp \overline{MR}$, $\overline{PR} \perp \overline{MR}$, $\overline{NM} \cong \overline{PR}$
Prove: $MO \cong OR$

7. Given: $DL$ bisects $BN$, $\angle XLN \cong \angle XDB$
Prove: $LN \cong DB$

Write a flow proof.
8. Given: $\overline{MN} \cong \overline{PQ}$, $\angle M \cong \angle Q$, $\angle 2 \cong \angle 3$
Prove: $\triangle MLP \cong \triangle QLN$

9. Given: $DE \parallel JK$, $DK$ bisects $JE$. $\overline{DE} \cong \overline{JK}$
Prove: $\triangle EGD \cong \triangle JGK$

GARDENING For Exercises 10 and 11, use the following information.
Beth is planning a garden. She wants the triangular sections $\triangle CFD$ and $\triangle HFG$ to be congruent. $F$ is the midpoint of $DG$, and $DG = 16$ feet.

10. Suppose $CD$ and $GH$ each measure 4 feet and the measure of $\angle CFD$ is 29. Determine whether $\triangle CFD \cong \triangle HFG$. Justify your answer.
11. Suppose $F$ is the midpoint of $CH$, and $CH \cong DG$. Determine whether $\triangle CFD \cong \triangle HFG$. Justify your answer.

Write a flow proof.
12. Given: $\angle V \cong \angle S$, $TV \cong QS$
Prove: $VR \cong SR$

13. Given: $EF \parallel FK$, $\overline{FG} \parallel \overline{KH}$, $\overline{EF} \cong \overline{GH}$
Prove: $\triangle EFG \cong \triangle FKH$

Write a paragraph proof.
14. Given: $\angle F \cong \angle J$, $\angle E \cong \angle H$, $\overline{EC} \cong \overline{GH}$
Prove: $\overline{EF} \cong \overline{HJ}$

15. Given: $\overline{TX} \parallel \overline{SY}$, $\angle TXY \cong \angle TSY$
Prove: $\triangle TSY \cong \triangle YXT$
PROOF Write a two-column proof.

16. Given: \( \angle MYT \cong \angle NYT, \)
\( \angle MTY \cong \angle NTY \)
Prove: \( \triangle RYM \cong \triangle RYN \)

17. Given: \( \triangle BMI \cong \triangle KMT, \)
\( IP \cong PT \)
Prove: \( \triangle IPK \cong \triangle TPB \)

KITES For Exercises 18 and 19, use the following information.
Austin is making a kite. Suppose \( JL \) is two feet, \( JM \) is 2.7 feet, and the measure of \( \angle NJM \) is 68.

18. If \( N \) is the midpoint of \( JL \) and \( KM \perp JL \), determine whether \( \triangle JKN \cong \triangle LKN \). Justify your answer.

19. If \( \overline{JM} \cong \overline{LM} \) and \( \angle NJM \cong \angle NLM \), determine whether \( \triangle JNM \cong \triangle LNM \). Justify your answer.

Complete each congruence statement and the postulate or theorem that applies.

20. If \( \overline{IM} \cong \overline{RV} \) and \( \angle 2 \cong \angle 5 \), then \( \triangle INM \cong \triangle \) by ?.

21. If \( \overline{IR} \parallel \overline{MV} \) and \( \overline{IR} \cong \overline{MV} \), then \( \triangle IRN \cong \triangle \) by ?.

22. Which One Doesn’t Belong? Identify the term that does not belong with the others. Explain your reasoning.

   ASA  SSS  SSA  AAS

23. REASONING Find a counterexample to show why AAA (Angle-Angle-Angle) cannot be used to prove congruence in triangles.

24. OPEN ENDED Draw and label two triangles that could be proved congruent by SAS.

25. CHALLENGE Neva wants to estimate the distance between herself and a toy boat. She adjusts the visor of her cap so that it is in line with her line of sight to the toy boat. She keeps her neck stiff and turns her body to establish a line of sight to a point on the ground. Then she paces out the distance to the new point. Is the distance from the toy boat the same as the distance she just paced out? Explain your reasoning.

26. Writing in Math Use the information about construction on page 234 to explain how congruent triangles are used in construction. Include why it is important to use congruent triangles for support.
27. Given: \( \overline{BC} \) is perpendicular to \( \overline{AD} \); \( \angle 1 \cong \angle 2 \).

Which theorem or postulate could be used to prove \( \triangle ABC \cong \triangle DBC \)?

- A AAS  
- B ASA  
- C SAS  
- D SSS

28. REVIEW Which expression can be used to find the values of \( s(n) \) in the table?

<table>
<thead>
<tr>
<th>( n )</th>
<th>( -8 )</th>
<th>( -4 )</th>
<th>( -1 )</th>
<th>( 0 )</th>
<th>( 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s(n) )</td>
<td>1.00</td>
<td>2.00</td>
<td>2.75</td>
<td>3.00</td>
<td>3.25</td>
</tr>
</tbody>
</table>

- F \( -2n + 3 \)  
- G \( -n + 7 \)  
- H \( \frac{1}{4}n + 3 \)  
- J \( \frac{1}{2}n + 5 \)

Write a flow proof. (Lesson 4-4)

29. Given: \( \overline{BA} \cong \overline{DE}, \overline{DA} \cong \overline{BE} \)

Prove: \( \triangle BEA \cong \triangle DAE \)

30. Given: \( \overline{XZ} \perp \overline{WY}, \overline{XZ} \) bisects \( \overline{WY} \).

Prove: \( \triangle WZX \cong \triangle YZX \)

Verify congruence and name the congruence transformation. (Lesson 4-3)

31. \( \triangle RTS \cong \triangle R'T'S' \)

32. \( \triangle MNP \cong \triangle M'N'P' \)

Write each statement in if-then form. (Lesson 2-3)

33. Happy people rarely correct their faults.

34. A champion is afraid of losing.

PREREQUISITE SKILL Classify each triangle according to its sides. (Lesson 4-1)

35.  
36.  
37.  

Lesson 4-5 Proving Congruence—ASA, AAS 241
In Lessons 4-4 and 4-5, you learned theorems and postulates to prove triangles congruent. Do these theorems and postulates apply to right triangles?

**ACTIVITY 1  Triangle Congruence**

Study each pair of right triangles.

\[ \text{a. } \]
\[ \text{b. } \]
\[ \text{c. } \]

**ANALYZE THE RESULTS**

1. Is each pair of triangles congruent? If so, which congruence theorem or postulate applies?

2. Rewrite the congruence rules from Exercise 1 using leg, (L), or hypotenuse, (H), to replace side. Omit the A for any right angle since we know that all right triangles contain a right angle and all right angles are congruent.

3. **MAKE A CONJECTURE** If you know that the corresponding legs of two right triangles are congruent, what other information do you need to declare the triangles congruent? Explain.

In Lesson 4-5, you learned that SSA is not a valid test for determining triangle congruence. Can SSA be used to prove right triangles congruent?

**ACTIVITY 2  SSA and Right Triangles**

How many right triangles exist that have a hypotenuse of 10 centimeters and a leg of 7 centimeters?

**Step 1**
Draw \( \overline{XY} \) so that \( XY = 7 \) centimeters.

**Step 2**
Use a protractor to draw a ray from \( Y \) that is perpendicular to \( \overline{XY} \).

**Step 3**
Open your compass to a width of 10 centimeters. Place the point at \( X \) and draw a long arc to intersect the ray.

**Step 4**
Label the intersection \( Z \) and draw \( \overline{XZ} \) to complete \( \triangle XYZ \).
ANALYZE THE RESULTS

4. Does the model yield a unique triangle?
5. Can you use the lengths of the hypotenuse and a leg to show right triangles are congruent?
6. Make a conjecture about the case of SSA that exists for right triangles.

The two activities provide evidence for four ways to prove right triangles congruent.

### KEY CONCEPT

<table>
<thead>
<tr>
<th>Theorems</th>
<th>Abbreviation</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>4.6 Leg-Leg Congruence</strong> If the legs of one right triangle are congruent to the corresponding legs of another right triangle, then the triangles are congruent.</td>
<td>LL</td>
<td><img src="Image" alt="Example" /></td>
</tr>
<tr>
<td><strong>4.7 Hypotenuse-Angle Congruence</strong> If the hypotenuse and acute angle of one right triangle are congruent to the hypotenuse and corresponding acute angle of another right triangle, then the two triangles are congruent.</td>
<td>HA</td>
<td><img src="Image" alt="Example" /></td>
</tr>
<tr>
<td><strong>4.8 Leg-Angle Congruence</strong> If one leg and an acute angle of one right triangle are congruent to the corresponding leg and acute angle of another right triangle, then the triangles are congruent.</td>
<td>LA</td>
<td><img src="Image" alt="Example" /></td>
</tr>
<tr>
<td><strong>Postulate</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>4.4 Hypotenuse-Leg Congruence</strong> If the hypotenuse and a leg of one right triangle are congruent to the hypotenuse and corresponding leg of another right triangle, then the triangles are congruent.</td>
<td>HL</td>
<td><img src="Image" alt="Example" /></td>
</tr>
</tbody>
</table>

### Exercises

**PROOF** Write a paragraph proof of each theorem.

7. Theorem 4.6
8. Theorem 4.7
9. Theorem 4.8 (Hint: There are two possible cases.)

Use the figure to write a two-column proof.

10. Given: $\overline{ML} \perp \overline{MK}, \overline{JK} \perp \overline{KM}$
    $\angle J \cong \angle L$
    Prove: $\overline{JM} \cong \overline{KL}$

11. Given: $\overline{JK} \perp \overline{KM}, \overline{JM} \cong \overline{KL}$
    $\overline{ML} \parallel \overline{JK}$
    Prove: $\overline{ML} \cong \overline{JK}$
The art of Lois Mailou Jones, a twentieth-century artist, includes paintings and textile design, as well as book illustration. Notice the isosceles triangles in this painting, *Damballah*.

**Properties of Isosceles Triangles** In Lesson 4-1, you learned that isosceles triangles have two congruent sides. Like the right triangle, the parts of an isosceles triangle have special names.

- **Vertex Angle**: The angle formed by the congruent sides is called the vertex angle.
- **Base Angles**: The two angles formed by the base and one of the congruent sides are called base angles.

**GEOMETRY LAB**

**Isosceles Triangles**

**MODEL**
- Draw an acute triangle on patty paper with $AC \cong BC$.
- Fold the triangle through $C$ so that $A$ and $B$ coincide.

**ANALYZE**
1. What do you observe about $\angle A$ and $\angle B$?
2. Draw an obtuse isosceles triangle. Compare the base angles.
3. Draw a right isosceles triangle. Compare the base angles.

The results of the Geometry Lab suggest Theorem 4.9.
EXAMPLE Proof of Theorem

Write a two-column proof of the Isosceles Triangle Theorem.

**Given:** \( \angle PQR, \overline{PQ} \cong \overline{RQ} \)

**Prove:** \( \angle P \cong \angle R \)

**Proof:**

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Let ( S ) be the midpoint of ( \overline{PR} ).</td>
<td>1. Every segment has exactly one midpoint.</td>
</tr>
<tr>
<td>2. Draw an auxiliary segment ( \overline{QS} )</td>
<td>2. Two points determine a line.</td>
</tr>
<tr>
<td>3. ( \overline{PS} \cong \overline{RS} )</td>
<td>3. Midpoint Theorem</td>
</tr>
<tr>
<td>4. ( \overline{QS} \cong \overline{QS} )</td>
<td>4. Congruence of segments is reflexive.</td>
</tr>
<tr>
<td>5. ( \overline{PQ} \cong \overline{RQ} )</td>
<td>5. Given</td>
</tr>
<tr>
<td>6. ( \triangle PQS \cong \triangle RQS )</td>
<td>6. SSS</td>
</tr>
<tr>
<td>7. ( \angle P \cong \angle R )</td>
<td>7. CPCTC</td>
</tr>
</tbody>
</table>

**CHECK Your Progress**

1. Write a two-column proof.

   **Given:** \( \overline{CA} \cong \overline{BC}; \overline{KC} \cong \overline{CJ} \)

   \( C \) is the midpoint of \( \overline{BK} \).

   **Prove:** \( \triangle ABC \cong \triangle JKC \)

**Test-Taking Tip**

Diagrams Label the diagram with the given information. Use your drawing to plan the next step in solving the problem.

**STANDARDIZED TEST EXAMPLE**

Find a Missing Angle Measure

If \( \overline{GH} \cong \overline{HK}, \overline{HJ} \cong \overline{JK}, \) and \( m\angle GJK = 100, \) what is \( m\angle HGK? \)

A 10 \hspace{1cm} B 15 \hspace{1cm} C 20 \hspace{1cm} D 25

**Read the Test Item**

\( \triangle GHK \) is isosceles with base \( \overline{GK}. \) Likewise, \( \triangle HJK \) is isosceles with base \( \overline{HK}. \)

(continued on the next page)
**Solve the Test Item**

**Step 1**  The base angles of $\triangle HJK$ are congruent. Let $x = m\angle KHJ = m\angle HKJ$.

$$m\angle KHJ + m\angle HKJ + m\angle HJK = 180$$  \hspace{1em} \text{Angle Sum Theorem}

$$x + x + 100 = 180$$  \hspace{1em} \text{Substitution}

$$2x + 100 = 180$$  \hspace{1em} \text{Add.}

$$2x = 80$$  \hspace{1em} \text{Subtract 100 from each side.}

$$x = 40$$  \hspace{1em} \text{So, } m\angle KHJ = m\angle HKJ = 40.

**Step 2**  $\angle GHK$ and $\angle KHJ$ form a linear pair. Solve for $m\angle GHK$.

$$m\angle KHJ + m\angle GHK = 180$$  \hspace{1em} \text{Linear pairs are supplementary.}

$$40 + m\angle GHK = 180$$  \hspace{1em} \text{Substitution}

$$m\angle GHK = 140$$  \hspace{1em} \text{Subtract 40 from each side.}

**Step 3**  The base angles of $\triangle GHK$ are congruent. Let $y$ represent $m\angle HGK$ and $m\angle GKH$.

$$m\angle GHK + m\angle HGK + m\angle GKH = 180$$  \hspace{1em} \text{Angle Sum Theorem}

$$140 + y + y = 180$$  \hspace{1em} \text{Substitution}

$$140 + 2y = 180$$  \hspace{1em} \text{Add.}

$$2y = 40$$  \hspace{1em} \text{Subtract 140 from each side.}

$$y = 20$$  \hspace{1em} \text{Divide each side by 2.}

The measure of $\angle HGK$ is 20. Choice C is correct.

**2.** $\triangle ABD$ is isosceles, and $\triangle ACD$ is a right triangle. If $m\angle 6 = 136$, what is $m\angle 3$?

- F 21
- G 37
- H 68
- J 113

The converse of the Isosceles Triangle Theorem is also true.

**THEOREM 4.10**

If two angles of a triangle are congruent, then the sides opposite those angles are congruent.

**Abbreviation:** Conv. of Isos. $\triangle$ Th.

**Example:** If $\angle D \cong \angle F$, then $\overline{DE} \cong \overline{FE}$.

You will prove Theorem 4.10 in Exercise 13.

**Look Back**

You can review converses in Lesson 2-3.

**Personal Tutor at geometryonline.com**
A triangle is equilateral if and only if it is equiangular.

Each angle of an equilateral triangle measures 60°.

**EXAMPLE**

**Congruent Segments and Angles**

**a. Name two congruent angles.**

\[ \angle AFC \text{ is opposite } \overline{AC} \text{ and } \angle ACF \text{ is opposite } \overline{AF}, \]

\[ \therefore \angle AFC \cong \angle ACF. \]

**b. Name two congruent segments.**

By the converse of the Isosceles Triangle Theorem, the sides opposite congruent angles are congruent. So, \( \overline{BC} \cong \overline{BF} \).

**CHECK Your Progress**

3A. Name two congruent angles.

3B. Name two congruent segments.

**Properties of Equilateral Triangles**

Recall that an equilateral triangle has three congruent sides. The Isosceles Triangle Theorem leads to two corollaries about the angles of an equilateral triangle.

**Corollaries**

<table>
<thead>
<tr>
<th>4.3 A triangle is equilateral if and only if it is equiangular.</th>
<th>4.4 Each angle of an equilateral triangle measures 60°.</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Equilateral Triangle" /></td>
<td><img src="image" alt="Equilateral Triangle with Angle Measurements" /></td>
</tr>
</tbody>
</table>

You will prove Corollaries 4.3 and 4.4 in Exercises 11 and 12.

**EXAMPLE**

Use Properties of Equilateral Triangles

\( \triangle EFG \) is equilateral, and \( \overline{EH} \) bisects \( \angle E \).

**a. Find \( m\angle 1 \) and \( m\angle 2 \).**

Each angle of an equilateral triangle measures 60°.

So, \( m\angle 1 + m\angle 2 = 60 \). Since the angle was bisected, \( m\angle 1 = m\angle 2 \). Thus, \( m\angle 1 = m\angle 2 = 30 \).

**b. ALGEBRA**

Find \( x \).

\[ m\angle EFH + m\angle 1 + m\angle EHF = 180 \quad \text{Angle Sum Theorem} \]

\[ 60 + 30 + 15x = 180 \]

\[ 90 + 15x = 180 \quad \text{Add.} \]

\[ 15x = 90 \quad \text{Subtract 90 from each side.} \]

\[ x = 6 \quad \text{Divide each side by 15.} \]
\[ \triangle DEF \text{ is equilateral}. \]

**4A.** Find \( x \).

**4B.** Find \( m\angle 1 \) and \( m\angle 2 \).

---

**Examples 1, 4** (pp. 245, 247)

**PROOF** Write a two-column proof.

1. **Given:** \( \triangle CTE \) is isosceles with vertex \( \angle C \).  
   \[ m\angle T = 60 \]  
   **Prove:** \( \triangle CTE \) is equilateral.

**Example 2** (p. 246)

2. **STANDARDIZED TEST PRACTICE** If \( \overline{PQ} \cong \overline{QS}, \overline{QR} \cong \overline{RS} \), and \( m\angle PRS = 72 \), what is \( m\angle QPS \)?
   
   A. 27  
   B. 54  
   C. 63  
   D. 72

**Example 3** (p. 247)

Refer to the figure.

3. If \( \overline{AD} \cong \overline{AH} \), name two congruent angles.

4. If \( \angle BDH \cong \angle BHD \), name two congruent segments.

---

**Exercises**

Refer to the figure for Exercises 5–10.

5. If \( \overline{LT} \cong \overline{LR} \), name two congruent angles.

6. If \( \overline{LX} \cong \overline{LW} \), name two congruent angles.

7. If \( \overline{SL} \cong \overline{QL} \), name two congruent angles.

8. If \( \angle LXY \cong \angle LYX \), name two congruent segments.

9. If \( \angle LSR \cong \angle LRS \), name two congruent segments.

10. If \( \angle LYW \cong \angle LWY \), name two congruent segments.

**PROOF** Write a two-column proof.

11. Corollary 4.3  
12. Corollary 4.4  
13. Theorem 4.10

**Triangle \( LMN \)** is equilateral, and \( \overline{MP} \) bisects \( \overline{LN} \).

14. Find \( x \) and \( y \).

15. Find the measure of each side.

---

**\( \triangle KLN \) and \( \triangle LMN \)** are isosceles and \( m\angle JKN = 130 \).

Find each measure.

16. \( m\angle LNM \)  
17. \( m\angle M \)  
18. \( m\angle LKN \)  
19. \( m\angle J \)
In the figure, $JM \cong PM$ and $ML \cong PL$.

20. If $m \angle PLJ = 34$, find $m \angle JPM$.

21. If $m \angle PLJ = 58$, find $m \angle PML$.

$\triangle DFG$ and $\triangle FGH$ are isosceles, $m \angle FDH = 28$, and $DG \cong FG \cong FH$. Find each measure.

22. $m \angle DFG$

23. $m \angle DGF$

24. $m \angle FGH$

25. $m \angle GFH$

In the figure, $\overline{GK} \cong \overline{GH}$ and $\overline{HK} \cong \overline{KJ}$.

26. If $m \angle HGK = 28$, find $m \angle HKJ$.

27. If $m \angle HGK = 42$, find $m \angle HKJ$.

**PROOF** Write a two-column proof for each of the following.

28. **Given:** $\triangle XKF$ is equilateral. $XJ$ bisects $\angle X$. $N$ is the midpoint of $MP$.

29. **Given:** $\triangle MLP$ is isosceles. $N$ is the midpoint of $MP$.

**Prove:** $J$ is the midpoint of $KF$. $LN \perp MP$.

30. **DESIGN** The exterior of Spaceship Earth at Epcot Center in Orlando, Florida, is made up of triangles. Describe the minimum requirement to show that these triangles are equilateral.

**ALGEBRA** Find $x$.

31. 

32. 

33. 

34. **OPEN ENDED** Describe a method to construct an equilateral triangle.

35. **CHALLENGE** In the figure, $\triangle ABC$ is isosceles, $\triangle DCE$ is equilateral, and $\triangle FCG$ is isosceles. Find the measures of the five numbered angles at vertex $C$.

36. **Writing in Math** Explain how triangles can be used in art. Describe at least three other geometric shapes and how they are used in art. Include an interpretation of how and why isosceles triangles are used in the painting shown at the beginning of the lesson.
37. In the figure below, $AE$ and $BD$ bisect each other at point $C$.

Which additional piece of information would be enough to prove that $\overline{CD} \cong \overline{DE}$?
A $\angle A \cong \angle C$
B $\angle B \cong \angle D$
C $\angle ACB \cong \angle EDC$
D $\angle A \cong \angle B$

38. REVIEW  What quantity should be added to both sides of this equation to complete the square?
\[ x^2 - 10x = 3 \]
F $-25$
G $-5$
H $5$
J $25$

PROOF Write a paragraph proof.  (Lesson 4-5)

39. Given: $\angle N \cong \angle D$, $\angle G \cong \angle I$, $AN \cong SD$
Prove: $\triangle ANG \cong \triangle SDI$

40. Given: $\overline{VR} \perp \overline{RS}$, $\overline{UT} \perp \overline{SU}$
Prove: $\triangle VRS \cong \triangle TUS$

Determine whether $\triangle QRS \cong \triangle EGH$ given the coordinates of the vertices. Explain.  (Lesson 4-4)

41. $Q(-3, 1), R(1, 2), S(-1, -2), E(6, -2), G(2, -3), H(4, 1)$
42. $Q(1, -5), R(5, 1), S(4, 0), E(-4, -3), G(-1, 2), H(2, 1)$

43. LANDSCAPING  Lucas is drawing plans for a client’s backyard on graph paper. The client wants two perpendicular pathways to cross at the center of her backyard. If the center of the backyard is set at $(0, 0)$ and the first path goes from one corner of the backyard at $(-6, 12)$ to the other corner at $(6, -12)$, at what coordinates will the second path begin and end?  (Lesson 3-3)

Construct a truth table for each compound statement.  (Lesson 2-2)

44. $a$ and $b$
45. $\neg p$ or $\neg q$
46. $k$ and $\neg m$
47. $\neg y$ or $z$

GET READY for the Next Lesson

PREREQUISITE SKILL  Find the coordinates of the midpoint of the segment with endpoints that are given.  (Lesson 1-3)

48. $A(2, 15), B(7, 9)$
49. $C(-4, 6), D(2, -12)$
50. $E(3, 2.5), F(7.5, 4)$
Navigators developed a series of circles to create a coordinate grid that allows them to determine where they are on Earth. Similar to points in coordinate geometry, locations on this grid are given two values: an east/west value (longitude) and a north/south value (latitude).

Position and Label Triangles  Same as working with longitude and latitude, knowing the coordinates of points on a figure allows you to draw conclusions about it. Coordinate proof uses figures in the coordinate plane and algebra to prove geometric concepts. The first step in a coordinate proof is placing the figure on the coordinate plane.

**Position and Label a Triangle**

Position and label isosceles triangle $JKL$ on a coordinate plane so that base $JK$ is $a$ units long.

- Use the origin as vertex $J$ of the triangle.
- Place the base of the triangle along the positive $x$-axis.
- Position the triangle in the first quadrant.
- Since $K$ is on the $x$-axis, its $y$-coordinate is 0. Its $x$-coordinate is $a$ because the base is $a$ units long.
- $\triangle JKL$ is isosceles, so the $x$-coordinate of $L$ is halfway between 0 and $a$ or $\frac{a}{2}$. We cannot write the $y$-coordinate in terms of $a$, so call it $b$.

1. Position and label right triangle $HIJ$ with legs $\overline{HI}$ and $\overline{IJ}$ on a coordinate plane so that $\overline{HI}$ is $a$ units long and $\overline{IJ}$ is $b$ units long.
EXAMPLE

Find the Missing Coordinates

2. Name the missing coordinates of isosceles right triangle $EFG$.

Vertex $F$ is positioned at the origin; its coordinates are $(0, 0)$. Vertex $E$ is on the $y$-axis, and vertex $G$ is on the $x$-axis. So $\angle EFG$ is a right angle. Since $\triangle EFG$ is isosceles, $EF \cong GF$. $EF$ is $a$ units and $GF$ must be the same. So, the coordinates of $G$ are $(a, 0)$.

CHECK Your Progress

2. Name the missing coordinates of isosceles triangle $PDQ$.

Write Proofs After a figure is placed on the coordinate plane and labeled, we can coordinate proof to verify properties and to prove theorems.

EXAMPLE

Coordinate Proof

3. Write a coordinate proof to prove that the measure of the segment that joins the vertex of the right angle in a right triangle to the midpoint of the hypotenuse is one-half the measure of the hypotenuse.

Place the right angle at the origin and label it $A$. Use coordinates that are multiples of 2 because the Midpoint Formula takes half the sum of the coordinates.

Given: right $\triangle ABC$ with right $\angle BAC$

$P$ is the midpoint of $\overline{BC}$.

Prove: $AP = \frac{1}{2}BC$

Proof:

By the Midpoint Formula, the coordinates of $P$ are \left(\frac{0 + 2c}{2}, \frac{-2b + 0}{2}\right)$ or $(c, b)$.

Use the Distance Formula to find $AP$ and $BC$.

$$AP = \sqrt{(c - 0)^2 + (b - 0)^2} = \sqrt{c^2 + b^2}$$

$$BC = \sqrt{(2c - 0)^2 + (0 - 2b)^2} = \sqrt{4c^2 + 4b^2} \text{ or } 2\sqrt{c^2 + b^2}$$

$$\frac{1}{2}BC = \sqrt{c^2 + b^2}$$

Therefore, $AP = \frac{1}{2}BC$.

CHECK Your Progress

3. Use a coordinate proof to show that the triangles shown are congruent.

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ARROWHEADS Write a coordinate proof to prove that this arrowhead is shaped like an isosceles triangle. The arrowhead is 3 inches long and 1.5 inches wide.

The first step is to label the coordinates of each vertex. Q is at the origin, and T is at (1.5, 0). The y-coordinate of R is 3. The x-coordinate is halfway between 0 and 1.5 or 0.75. So, the coordinates of R are (0.75, 3).

If the legs of the triangle are the same length, it is isosceles. Use the Distance Formula to find \( QR \) and \( RT \).

\[
QR = \sqrt{(0.75 - 0)^2 + (3 - 0)^2} = \sqrt{0.5625 + 9} = \sqrt{9.5625}
\]

\[
RT = \sqrt{(1.5 - 0.75)^2 + (0 - 3)^2} = \sqrt{0.5625 + 9} = \sqrt{9.5625}
\]

Since each leg is the same length, \( \triangle QRT \) is isosceles. The arrowhead is shaped like an isosceles triangle.

4. Use coordinate geometry to classify a triangle with vertices located at the following coordinates \( A(0, 0), B(0, 6) \), and \( C(3, 3) \).

Example 1
(p. 251)

Position and label each triangle on the coordinate plane.
1. isosceles \( \triangle FGH \) with base \( FH \) that is 2\( b \) units long
2. equilateral \( \triangle CDE \) with sides \( a \) units long

Example 2
(p. 252)

Name the missing coordinates of each triangle.
3. \[ P(?, ?) \]
4. \[ P(0, c) \]

Example 3
(p. 252)

5. Write a coordinate proof for the following statement. The midpoint of the hypotenuse of a right triangle is equidistant from each of the vertices.

Example 4
(p. 253)

6. FLAGS Write a coordinate proof to prove that the large triangle in the center of the flag is isosceles. The dimensions of the flag are 4 feet by 6 feet, and point \( B \) of the triangle bisects the bottom of the flag.
Position and label each triangle on the coordinate plane.
7. isosceles $\triangle QRT$ with base $\overline{QR}$ that is $b$ units long
8. equilateral $\triangle MNP$ with sides $2a$ units long
9. isosceles right $\triangle JML$ with hypotenuse $\overline{JM}$ and legs $c$ units long
10. equilateral $\triangle WXZ$ with sides $\frac{1}{2}b$ units long
11. isosceles $\triangle PWY$ with base $\overline{PW}(a + b)$ units long
12. right $\triangle XYZ$ with hypotenuse $\overline{XZ}$, the length of $\overline{ZY}$ is twice $XY$, and $\overline{XY}$ is $b$ units long

Name the missing coordinates of each triangle.
13. $P(0, 0)$ $Q(2a, 0)$ $R(?, b)$
14. $Q(2a, 0)$ $P(?, ?)$ $L(0, 0)$
15. $N(?, ?)$ $P(?, ?)$ $K(2a, 0)$
16. $F(b, b\sqrt{3})$ $D(?, ?)$ $O(C(0, 0), D(?, ?))$
17. $E(?, ?)$ $B(?, ?)$ $O(C(a, 0), E(?, ?))$
18. $P(?, ?)$ $M(-2b, 0)$ $N(?, ?)$

Write a coordinate proof for each statement.
19. The segments joining the vertices of the base angles to the midpoints of the legs of an isosceles triangle are congruent.
20. The three segments joining the midpoints of the sides of an isosceles triangle form another isosceles triangle.
21. If a line segment joins the midpoints of two sides of a triangle, then it is parallel to the third side.
22. If a line segment joins the midpoints of two sides of a triangle, then its length is equal to one-half the length of the third side.

**NAVIGATION** For Exercises 23 and 24, use the following information.
A motor boat is located 800 yards from the port. There is a ship 800 yards to the east and another ship 800 yards to the north of the motor boat.
23. Write a coordinate proof to prove that the port, motor boat, and the ship to the north form an isosceles right triangle.
24. Write a coordinate proof to prove that the distance between the two ships is the same as the distance from the port to the northern ship.

**HIKING** For Exercises 25 and 26, use the following information.
Tami and Juan are hiking. Tami hikes 300 feet east of the camp and then hikes 500 feet north. Juan hikes 500 feet west of the camp and then 300 feet north.
25. Prove that Juan, Tami, and the camp form a right triangle.
26. Find the distance between Tami and Juan.
27. **STEEPLECHASE** Write a coordinate proof to prove that the triangles $ABD$ and $FBD$ are congruent. Suppose the hurdle is 6 feet wide and 4 feet tall, with the lower bar 1 foot off the ground.

Find the coordinates of point $C$ so $\triangle ABC$ is the indicated type of triangle. Point $A$ has coordinates $(0, 0)$ and $B$ has coordinates $(a, b)$.

- **28.** right triangle
- **29.** isosceles triangle
- **30.** scalene triangle

31. **OPEN ENDED** Draw a scalene right triangle on the coordinate plane so it simplifies a coordinate proof. Label the coordinates of each vertex. Explain why you placed the triangle this way.

32. **CHALLENGE** Classify $\triangle ABC$ by its angles and its sides. Explain.

33. **Writing in Math** Use the information about the coordinate plane given on page 251 to explain how the coordinate plane can be used in proofs. Include a list of the different types of proof and a theorem from the chapter that could be proved using a coordinate proof.

34. **Extra Practice**

35. **Review** What is the $x$-coordinate of the solution to the system of equations shown below?

$$
\begin{align*}
2x - 3y &= 3 \\
-4x + 2y &= -18
\end{align*}
$$

- **36.** Given: $\angle 3 \cong \angle 4$
  Prove: $QR \cong QS$

- **37.** Given: isosceles triangle $\triangle JKN$ with vertex $\angle N$, $JK \parallel LM$
  Prove: $\triangle NML$ is isosceles.

- **38.** Given: $AD \cong CE$; $AD \parallel CE$
  Prove: $\triangle ABD \cong \triangle EBC$

39. **JOBS** A studio engineer charges a flat fee of $450 for equipment rental and $42 an hour for recording and mixing time. Write the equation that shows the cost to hire the studio engineer as a function of time. How much would it cost to hire the studio engineer for 17 hours? *(Lesson 3-4)*
**Key Concepts**

**Classifying Triangles** (Lesson 4-1)
- Triangles can be classified by their angles as acute, obtuse, or right.
- Triangles can be classified by their sides as scalene, isosceles, or equilateral.

**Angles of Triangles** (Lesson 4-2)
- The sum of the measures of the angles of a triangle is 180°.
- The measures of an exterior angle are equal to the sum of the measures of the two remote interior angles.

**Congruent Triangles** (Lessons 4-3 through 4-5)
- If all of the corresponding sides of two triangles are congruent, then the triangles are congruent (SSS).
- If two corresponding sides of two triangles and the included angle are congruent, then the triangles are congruent (SAS).
- If two pairs of corresponding angles and the included sides of two triangles are congruent, then the triangles are congruent (ASA).
- If two pairs of corresponding angles and a pair of corresponding, nonincluded sides of two triangles are congruent, then the triangles are congruent (AAS).

**Isosceles Triangles** (Lesson 4-6)
- A triangle is equilateral if and only if it is equiangular.

**Triangles and Coordinate Proof** (Lesson 4-7)
- Coordinate proofs use algebra to prove geometric concepts.
- The Distance Formula, Slope Formula, and Midpoint Formula are often used in coordinate proof.

**Key Vocabulary**

- acute triangle (p. 202)
- base angles (p. 244)
- congruence transformation (p. 219)
- congruent triangles (p. 217)
- coordinate proof (p. 251)
- corollary (p. 213)
- equiangular triangle (p. 202)
- equilateral triangle (p. 203)
- exterior angle (p. 211)
- flow proof (p. 212)
- included side (p. 234)
- isosceles triangle (p. 203)
- obtuse triangle (p. 202)
- remote interior angles (p. 211)
- right triangle (p. 202)
- scalene triangle (p. 203)
- vertex angle (p. 244)

**Vocabulary Check**
Select the word from the list above that best completes the following statements.

1. A triangle with an angle measure greater than 90° is a(n) _____?
2. A triangle with exactly two congruent sides is a(n) _____?
3. A triangle that has an angle with a measure of exactly 90° is a(n) _____?
4. An equiangular triangle is a form of a(n) _____?
5. A(n) _____?_____ uses figures in the coordinate plane and algebra to prove geometric concepts.
6. A(n) _____?_____ preserves a geometric figure’s size and shape.
7. If all corresponding sides and angles of two triangles are congruent, those triangles are _____?_____.
Lesson-by-Lesson Review

4-1 Classifying Triangles (pp. 202–208)

Classify each triangle by its angles and by its sides if \( m\angle ABC = 100 \).

- 8. \( \triangle ABC \)
- 9. \( \triangle BDP \)
- 10. \( \triangle BPQ \)

Example 1 Find the measures of the sides of \( \triangle TUV \). Classify the triangle by sides.

Use the Distance Formula to find the measure of each side.

\[
TU = \sqrt{(-5 - (-2))^2 + (4 - (-2))^2} = \sqrt{9 + 36} = \sqrt{45}
\]
\[
UV = \sqrt{(3 - (-5))^2 + (1 - 4)^2} = \sqrt{64 + 9} = \sqrt{73}
\]
\[
VT = \sqrt{(-2 - 3)^2 + (-2 - 1)^2} = \sqrt{25 + 9} = \sqrt{34}
\]

Since the measures of the sides are all different, the triangle is scalene.

4-2 Angles of Triangles (pp. 210–216)

Find each measure.

- 12. \( m\angle 1 \)
- 13. \( m\angle 2 \)
- 14. \( m\angle 3 \)

Example 2 If \( \overline{TU} \perp \overline{UV} \) and \( \overline{UV} \perp \overline{VW} \), find \( m\angle 1 \).

Use the Angle Sum Theorem to write an equation.

\[
m\angle 1 + 72 + m\angle TVW = 180
\]
\[
m\angle 1 + 72 + (90 - 27) = 180
\]
\[
m\angle 1 + 135 = 180
\]
\[
m\angle 1 = 45
\]
**4-3 Congruent Triangles (pp. 217–223)**

Name the corresponding angles and sides for each pair of congruent triangles.

16. \( \triangle EFG \cong \triangle DCB \)

17. \( \triangle NCK \cong \triangle KER \)

18. **QUILTING** Meghan’s mom is going to enter a quilt at the state fair. Name the congruent triangles found in the quilt block.

Example 3 If \( \triangle EFG \cong \triangle JKL \), name the corresponding congruent angles and sides.

The letters of the triangles correspond to the congruent angles and sides. \( \angle E \cong \angle J \), \( \angle F \cong \angle K \), \( \angle G \cong \angle L \), \( EF \cong JK \), \( FG \cong KL \), and \( EG \cong JL \).

**4-4 Proving Congruence—SSS, SAS (pp. 225–232)**

Determine whether \( \triangle MNP \cong \triangle QRS \) given the coordinates of the vertices. Explain.

19. \( M(0, 3), N(-4, 3), P(-4, 6), Q(5, 6), R(2, 6), S(2, 2) \)

20. \( M(3, 2), N(7, 4), P(6, 6), Q(-2, 3), R(-4, 7), S(-6, 6) \)

21. **GAMES** In a game, Lupe’s boats are placed at coordinates \((3, 2), (0, -4), \) and \((6, -4)\). Do her ships form an equilateral triangle?

22. Triangle \( ABC \) is an isosceles triangle with \( AB \parallel BC \). If there exists a line \( BD \) that bisects \( \angle ABC \), show that \( \triangle ABD \cong \triangle CBD \).

Example 4 Determine whether \( \triangle ABC \cong \triangle TUV \). Explain.

\[
\begin{align*}
AB &= \sqrt{[-1 - (-2)]^2 + (1 - 0)^2} \\
&= \sqrt{1 + 1} \text{ or } \sqrt{2} \\
BC &= \sqrt{[0 - (-1)]^2 + (-1 - 1)^2} \\
&= \sqrt{1 + 4} \text{ or } \sqrt{5} \\
CA &= \sqrt{(-2 - 0)^2 + [0 - (-1)]^2} \\
&= \sqrt{4 + 1} \text{ or } \sqrt{5} \\
TU &= \sqrt{(3 - 4)^2 + (-1 - 0)^2} \\
&= \sqrt{1 + 1} \text{ or } \sqrt{2} \\
UV &= \sqrt{(2 - 3)^2 + [1 - (-1)]^2} \\
&= \sqrt{1 + 4} \text{ or } \sqrt{5} \\
VT &= \sqrt{(4 - 2)^2 + (0 - 1)^2} \\
&= \sqrt{4 + 1} \text{ or } \sqrt{5}
\end{align*}
\]

Therefore, \( \triangle ABC \cong \triangle TUV \) by SSS.
Mixed Problem Solving

For mixed problem-solving practice, see page 831.

Chapter 4
Study Guide and Review

4-6

Proving Congruence—ASA, AAS (pp. 234–241)

For Exercises 23 and 24, use the figure and write a two-column proof.

23. Given: \( DF \) bisects \( \angle CDE \).
\( CE \perp DF \)
Prove: \( \triangle DGC \cong \triangle DGE \)

24. Given: \( \triangle DGC \cong \triangle DGE \)
\( \triangle GCF \cong \triangle GEF \)
Prove: \( \triangle DFC \cong \triangle DFE \)

25. KITES Kyras kite is stuck in a set of power lines. If the power lines are stretched so that they are parallel with the ground, prove that \( \triangle ABD \cong \triangle CDB \).

Example 5 Write a proof.

Given: \( JK \parallel MN \)
\( L \) is the midpoint of \( KM \).
Prove: \( \triangle JLK \cong \triangle NLM \)

Flow Proof:

\[ \begin{align*}
\text{Given} & \quad JK \parallel MN \\
L & \text{ is the midpoint of } KM \\
\triangle JLK & \cong \triangle NLM \\
\text{Vert } \angle & \text{ are } =. \\
\triangle LJK & \cong \triangle LMN \\
\text{Alt. Int. Th.} & \\
\triangle KLM & \cong \triangle MLN \\
\text{Midpt. Th.} & \\
\triangle JLK \cong \triangle NLM & \\
\text{ASA} &
\end{align*} \]

Example 6 If \( \overline{FG} \cong \overline{GJ}, \overline{GJ} \cong \overline{JH}, \overline{FH} \cong \overline{FH}, \) and \( m\angle GJH = 40, \) find \( m\angle H. \)

\( \triangle GHJ \) is isosceles with base \( \overline{GH}, \) so \( \angle JGH \cong \angle H \) by the Isosceles Triangle Theorem. Thus, \( m\angle JGH = m\angle H. \)

\( m\angle GJH + m\angle JGH + m\angle H = 180 \)
\[ 40 + 2m\angle H = 180 \]
\[ 2 \cdot m\angle H = 140 \]
\[ m\angle H = 70 \]

Isosceles Triangles (pp. 244–250)

For Exercises 26–28, refer to the figure.

26. If \( PQ \parallel UQ \) and \( m\angle P = 32, \) find \( m\angle PUQ. \)

27. If \( RQ \parallel RS \) and \( m\angle RQS = 75, \) find \( m\angle R. \)

28. If \( RQ \parallel RS, RP \parallel RT, \) and \( m\angle RQS = 80, \) find \( m\angle P. \)

29. ART This geometric design from Western Cameroon uses approximations of isosceles triangles. Trace the figure. Identify and draw one isosceles triangle of each type from the design. Describe the similarities between the different triangles.
Position and label each triangle on the coordinate plane.

30. isosceles $\triangle TRI$ with base $\overline{TI} 4a$ units long

31. equilateral $\triangle BCD$ with side length $6m$ units long

32. right $\triangle JKL$ with leg lengths of $a$ units and $b$ units

33. BOATS A sailboat is located 400 meters to the east and 250 meters to the north of a dock. A canoe is located 400 meters to the west and 250 meters to the north of the same dock. Show that the sailboat, the canoe, and the dock all form an isosceles triangle.

Position and label isosceles right triangle $\triangle ABC$ with bases of length $a$ units on the coordinate plane.

- Use the origin as the vertex of $\triangle ABC$ that has the right angle.
- Place each of the bases along an axis, one on the x-axis and the other on the y-axis.
- Since $B$ is on the x-axis, its y-coordinate is 0. Its x-coordinate is $a$ because the leg of the triangle is $a$ units long.

Since $\triangle ABC$ is isosceles, $C$ should also be a distance of $a$ units from the origin. Its coordinates should be $(0, -a)$, since it is on the negative y-axis.
Identify the indicated triangles in the figure if $\overline{PB} \perp \overline{AD}$ and $\overline{PA} \cong \overline{PC}$.

1. obtuse
2. isosceles
3. right

Find the measure of each angle in the figure.

4. $m\angle1$
5. $m\angle2$
6. $m\angle3$

7. Write a flow proof.
   Given: $\triangle JKM \cong \triangle JNM$
   Prove: $\triangle JKL \cong \triangle JNL$

Name the corresponding angles and sides for each pair of congruent triangles.

8. $\triangle DEF \cong \triangle PQR$
9. $\triangle FMG \cong \triangle HNJ$
10. $\triangle XYZ \cong \triangle ZYX$

11. MULTIPLE CHOICE In $\triangle ABC$, $\overline{AD}$ and $\overline{DC}$ are angle bisectors and $m\angle B = 76$.

What is $m\angle ADC$?
   
   A 26  C 76
   B 52  D 128

12. Determine whether $\triangle JKL \cong \triangle MNP$ given $J(-1, -2)$, $K(2, -3)$, $L(3, 1)$, $M(-6, -7)$, $N(-2, 1)$, and $P(5, 3)$. Explain.

In the figure, $\overline{FJ} \cong \overline{FH}$ and $\overline{GF} \cong \overline{GH}$.

13. If $m\angle JFH = 34$, find $m\angle J$.

14. If $m\angle GHJ = 152$ and $m\angle G = 32$, find $m\angle JFH$.

15. LANDSCAPING A landscaper designed a garden shaped as shown in the figure. The landscaper has decided to place point $B$ 22 feet east of point $A$, point $C$ 44 feet east of point $A$, point $E$ 36 feet south of point $A$, and point $D$ 36 feet south of point $C$. The angles at points $A$ and $C$ are right angles. Prove that $\triangle ABE \cong \triangle CBD$.

16. MULTIPLE CHOICE In the figure, $\triangle FGH$ is a right triangle with hypotenuse $\overline{FH}$ and $GJ \parallel GH$.

What is $m\angle JGH$?

   F 104  H 56
   G 62  J 28
Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. Use the proof to answer the question below.
   **Given:** $AD \parallel BC$
   **Prove:** $\triangle ABD \cong \triangle CDB$

   **Statements** | **Reasons**
   --- | ---
   1. $AD \parallel BC$ | 1. Given
   2. $\angle ABD \cong \angle CDB$, $\angle ADB \cong \angle CBD$ | 2. Alternate Interior Angles Theorem
   3. $BD \cong DB$ | 3. Reflexive Property
   4. $\triangle ABD \cong \triangle CDB$ | 4. ?

   What reason can be used to prove the triangles are congruent?
   - A AAS
   - B ASA
   - C SAS
   - D SSS

2. The graph of $y = 2x - 5$ is shown at the right. How would the graph be different if the number 2 in the equation was replaced with a 4?
   - F parallel to the line shown, but shifted two units higher
   - G parallel to the line shown, but shifted two units lower
   - H have a steeper slope, but intercept the $y$-axis at the same point
   - J have a less steep slope, but intercept the $y$-axis at the same point

3. **GRIDDABLE** What is $m\angle 1$ in degrees?

4. In the figure below, $BC \cong EF$ and $\angle B \cong \angle E$.

   Which additional information would be enough to prove $\triangle ABC \cong \triangle DEF$?
   - A $\angle A \cong \angle D$
   - B $\overline{AC} \cong \overline{DF}$
   - C $\overline{BC} \cong \overline{DF}$
   - D $\overline{DE} \perp \overline{EF}$

5. The diagram shows square $DEFG$. Which statement could not be used to prove $\triangle DEG$ is a right triangle?
   - F $(EG)^2 = (DG)^2 + (DE)^2$
   - G Definition of a Square
   - H $(\text{slope } DE)(\text{slope } DG) = 1$
   - J $(\text{slope } DE)(\text{slope } DG) = -1$

6. **ALGEBRA** Which equation is equivalent to $4(y - 2) - 3(2y - 4) = 9$?
   - A $2y - 4 = 9$
   - B $-2y + 4 = 9$
   - C $10y - 20 = 9$
   - D $-2y - 4 = 9$
7. In the quadrilateral, which pair of segments can be established to be congruent to prove that $\overline{AC} \parallel \overline{FD}$?

F $\overline{AC} \cong \overline{FD}$  H $\overline{BC} \cong \overline{FE}$
G $\overline{AF} \cong \overline{CD}$  J $\overline{BF} \cong \overline{CE}$

8. Which of the following is the inverse of the statement “If it is raining, then Kamika carries an umbrella?”

A If Kamika carries an umbrella, then it is raining.
B If Kamika does not carry an umbrella, then it is not raining.
C If it is not raining, then Kamika carries an umbrella.
D If it is not raining, then Kamika does not carry an umbrella.

9. ALGEBRA Which of the following describes the line containing the points (2, 4) and (0, -2)?

F $y = -3x + 2$  H $y = \frac{1}{3}x - 2$
G $y = \frac{-1}{3}x - 4$  J $y = -3x + 2$

10. A 9-foot tree casts a shadow on the ground. The distance from the top of the tree to the end of the shadow is 12 feet. To the nearest foot, how long is the shadow?

A 7 ft  C 10 ft
B 8 ft  D 12 ft

11. In the following proof, what property justifies statement 3?

Given: $\overline{AC} \cong \overline{MN}$
Prove: $AB + BC = MN$

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $\overline{AC} \cong \overline{MN}$</td>
<td>1. Given</td>
</tr>
<tr>
<td>2. $AC = MN$</td>
<td>2. Def. of $\cong$ segments</td>
</tr>
<tr>
<td>3. $AC = AB + BC$</td>
<td>3. ?</td>
</tr>
<tr>
<td>4. $AC + BC = MN$</td>
<td>4. Substitution</td>
</tr>
</tbody>
</table>

F Definition of Midpoint
G Transitive Property
H Segment Addition Postulate
J Commutative Property

12. If $\angle ACD$ is a right angle, what is the relationship between $\angle ACF$ and $\angle DCF$?

A complementary angles
B congruent angles
C supplementary angles
D vertical angles

13. The measures of $\triangle ABC$ are $5x$, $4x - 1$, and $3x + 13$.

a. Draw a figure to illustrate $\triangle ABC$ and find the measure of each angle.
b. Prove $\triangle ABC$ is an isosceles triangle.